PREDICTION OF INITIATION AND GROWTH OF CRACKS IN COMPOSITES. COUPLED STRESS AND ENERGY CRITERION OF THE FINITE FRACTURE MECHANICS

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Abstract

Computational methods developed to predict crack onset and growth in composites are briefly reviewed first. Assuming crack advances by (possibly) finite steps, which is the basic assumption of Finite Fracture Mechanics (FFM), in opposite to the hypothesis of crack advance by infinitesimal steps adopted in classical Linear Elastic Fracture Mechanics (LEFM), the coupled stress and energy criterion requires that both stress and energy conditions are simultaneously fulfilled. A quite general formulation of the coupled stress and energy criterion of FFM is introduced. A few examples of applications of this coupled criterion to the prediction of damage initiation in form of cracks in composites are mentioned. Finally, a new formulation of this coupled criterion representing a generalization of LEFM is proposed.

1. Introduction

Development and computational implementations of procedure able to provide accurate and efficient predictions of damage initiation and propagation in composites under static, fatigue and impact loads will be crucial for their successful applications in future taking advantage of their full potential. In view of the fact that composites are piecewise homogeneous materials with interfaces at different scales (from nano- to macro-scale) and of different characteristics, nonclassical methods for characterization of fracture and damage proceses should be developed. An adequate modeling of composites will also require more realistic material models covering complex rheologies, as viscoleastic or viscoplastic, in those cases where linear elastic model is not sufficient. The future computational models should be able to characterize competition between different dissipative phenomena, as matrix plasticity and interface debonding, taking into account strain rates.

In the present work we will focus on a quite fundamental problem of damage initiation and propagation in the form of cracks in composites subjected to static loads. As classical Linear Elastic Fracture Mechanics (LEFM)is not able to provide reasonable predictions about crack onset at a stress concentrator (e.g., U-notch or open hole on macro-scale and fibre or particle interface at micro-scale), or at a weak singularity (e.g., multimaterial corner in an adhesively bonded lap joints on macro-scale), or about crack crossing or deflecting at an interface, new non-classical theoretical approaches to fracture and computational methods should be developed and tested by series of suitable experiments, possibly using some inverse methods to tune the applied models.

The motivation of the present work has been an enormous effort made in the development of several nonclassical approaches and computational methods to fracture in last decades. It is expected that these approaches and methods will be able to provide significantly better predictions about damage initiation and propagation in composites than the classical ones. Although a brief review will be given for several of these nonclassical approaches to model fracture phenomena in composites and other materials, this work is focused on one of these approaches, called Finite Fracture Mechanics (FFM), in which the traditional dilemma: *stress-strength versus energy-fracture toughness*, with reference to what is governing the fracture initiation and propagation is formulated in a very explicit and neat form. In particular, we will analyse and discuss in depth the so-called coupled stress and energy criterion of FFM, introducing its quite general formulation. Then some of its applications to characterize several fracture and failure mechanisms in composites will be shown. Finally, a reformulation of this coupled criterion of FFM resulting in a generalization of classical LEFM will be proposed.

2. Computational approaches and methods for crack initiation and propagation

Several computational methods which can be applied to analyse initiation and propagation of cracks in composites on different scales are gathered and some relevant recent contributions or review works are cited. The following list of methods is neither exhaustive nor the methods are totally independent, as there are many connections between them and possibilities for their combinations. General numerical methods like Finite Element Method (FEM) and Boundary Element Method (BEM) to solve Partial Differential Equations (PDEs) can be employed in several of the specific methods presented in the following list, whereas some others provide own numerical procedure not requiring a FEM or BEM code.

- Computational fracture mechanics (CFM) [1]
- Cohesive zone models (CZM) [2]
- Extended finite element models (XFEM) [3]
- Continuum damage mechanics (CDM) [4]
- Discrete element method (DEM) [5]
- Peridynamics (PD) [6]
- Mathematical approaches based on nonconvex energy minimization (NEM) [7, 8]
- Theory of critical distances and Finite fracture mechanics (TCD and FFM) [9, 10, 11, 12]

Some of the presented methods, namely CZM, XFEM, NEM and FFM, and their applications to composites have recently been reviewed in [13].

3. Coupled stress and energy criterion of FFM. A general formulation

3.1. A general formulation of the coupled criterion

The coupled stress and energy criterion of FFM assumes that onset of a crack having a finite length (2D)/area(3D) is allowed if both stress and energy criteria, each one representing a necessary but not sufficient condition, are simultaneously fulfilled. We also assume that the time scale of crack onset is much shortest than the time scale where the whole structural problem under consideration is solved, i.e. from the point of view of FFM the crack onset is instantaneous, cf. [14]. It worth mentioning that such instantaneous fracture has also been observed in some atomistic studies.

Stress criterion defines a condition on stresses at the potential new crack surface(s) of a finite (i.e. non-infinitesimal) extension evaluated prior to the considered crack onset or advance of an existing crack. Let this new crack surface (or curve in a 2D formulation) be in general described as $\Delta S_c \subset R^d$ (d=2 or 3, respectively, in 2D or 3D problems). ΔS_c can define one or several surface components (cracks) which could simultaneously appear. A quite general formulation of the stress condition for formation of a new crack surface $\Delta S_c(\mathbf{a})$ can be written as

$$f\left(\sigma_{ij}(\cdot), \Delta S_c\right) \geq \sigma_c, \tag{1}$$

where f is a homogeneous functional of degree 1 of the stress distribution σ_{ij} on ΔS_c, i.e. $f\left(\lambda\sigma_{ij}, \Delta S_c\right) = \lambda f\left(\sigma_{ij}, \Delta S_c\right)$ for $\lambda \geq 0$, σ_c is a characteristic material strength parameter, usually tensile strength. Obviously f may additionally depend on several other material strength parameters.

As in general ΔS_c is not known a priori, to achieve a mathematical formulation easily implementable in a computational code we assume that ΔS_c is parameterized by a suitably chosen set of real parameters $a_1, \ldots, a_m$ (describing, e.g, the extension, location and orientation (angle) of the new crack surface) gathered in the parameter vector $\mathbf{a}$. Hence, each particular configuration of a potential new crack surface corresponds to a value of this parameter vector from a feasible parameter region denoted as $A \subset R^m$, writing $\Delta S_c(\mathbf{a})$ with $\mathbf{a} \in A$.

Let us give a few examples in 2D, where onset of a new crack occupying a segment at x axis $\langle 0, a\rangle$ ($a > 0$) is assessed, then $\Delta S_c(a) = \langle 0, a\rangle$. Functional f for simple pointwise and average normal stress criteria, respectively, proposed by Leguillon [9] and Cornetti et al [10], takes the form

$$f(\sigma(\cdot), \langle 0, a\rangle) = \min_{x\in\langle 0,a\rangle} \sigma(x), \quad \text{and} \quad f(\sigma(\cdot), \langle 0, a\rangle) = \frac{1}{a}\int_0^a \sigma(x)\mathrm{d}x \tag{2}$$

while, a quite general mixed mode pointwise stress criterion could be be expressed through, cf. [15, 16],

$$f((\sigma, \tau)(\cdot), \langle 0, a\rangle) = \min_{x\in\langle 0,a\rangle} \sqrt[p]{\left\langle \operatorname{sgn}(\sigma(x))|\sigma(x)|^p + \left(\frac{|\tau(x)|}{\tau_c/\sigma_c}\right)^p \right\rangle_+} \tag{3}$$

where σ_c and τ_c, respectively, are strengths of material under tension and shear, and $p > 0$, where usually $p = 2$. $\langle\rangle_+$ denotes the positive part of a real number.

Considering, for the sake of simplicity, a proportional loading governed by an applied nominal stress σ_{nominal} and a characteristic material strength parameter σ_c, the above stress condition can be rewritten in the following form

$$\frac{\sigma_{\text{nom.}}}{\sigma_c} \geq s(\mathbf{a}) \stackrel{\text{def}}{=} \frac{1}{f\left(\hat{\sigma}_{ij}(\cdot), \Delta S_c(\mathbf{a})\right)} \tag{4}$$

where $\hat{\sigma}_{ij} = \sigma_{ij}/\sigma_{\text{nom.}}$ corresponds to a unit nominally applied stress, and the dimensionless function s is defined for $\mathbf{a} \in A$. Several examples of analytically or semi-analytically evaluated function $s(\mathbf{a})$ can be found in [17, 18, 15].

Incremental energy criterion. Considering two states of a solid before and after onset of new crack surface $\Delta S_c(\mathbf{a})$, respectively, denoted as 0- and 1-state. Let the potential energy Π be defined as the sum of the stored strain energy $\mathcal{E}$ and the potential energy of external loads. Defining an increment of the potential energy between these two states as $\Delta\Pi(\Delta S_c(\mathbf{a})) = \Pi_1 - \Pi_0$, it is given by the increment of stored strain energy minus the work of external loads during crack onset denoted as $\Delta W(\Delta S_c(\mathbf{a})$. Hence, $\Delta\Pi = \Delta\mathcal{E} - \Delta W$. ΔW vanishes under displacement control.

Let the energy dissipated associated to this abrupt formation of a new crack surface (maybe by several dissipative mechanisms as breaking bonds across the new crack surface, plastic and viscous deformations, friction, etc.) be defined as $\Delta\mathcal{R}(\Delta S_c(\mathbf{a}))$. The incremental energy balance leads to (assuming a static state with zero kinematic energy at 0-state and neglecting heat exchange) necessary incremental energy condition, cf. [9, 16],

$$-\Delta\Pi(\Delta S_c(\mathbf{a})) \geq \Delta\mathcal{R}(\Delta S_c(\mathbf{a})) \tag{5}$$

which means that the released energy due to new crack surface onset should be equal or larger than the dissipated energy. Other reinterpretation is that the sum $\Pi + \mathcal{R}$ ($\mathcal{R}$ meaning total dissipated energy during the load history) should keep constant or decrease at a crack onset.

How the released energy can be calculated? There are at least four ways used in literature [19, 16]:

- A usual procedure in the case of linear elastic material behaviour is to take advantage of the relation between ERR G and potential energy variation, due to Griffith, $G = -\frac{\partial\Pi}{\partial\Delta S_C}$ which implies that formally $-\Delta\Pi = \int_{\Delta S_C} G$. This is often the most efficient and accurate way, as G is for many configurations available in fracture mechanics handbooks and works or can be computed quite easily by commercial FEM codes.
- By incremental crack closure technique, which is a variant of the VCCT (associated to an infinitesimal crack extension, at leat theoretically). This is applicable not only to linear elastic case, but also to other material rheologies, as linear viscoelastic behaviour assuming an instantaneous crack onset during which no viscous dissipation can take place.
- By evaluation of the variation of displacements and tractions at outer solid boundaries [16], again available for linear elastic material behaviour.

- The most general procedure, working also for other material rheologies, as elastoplastic, is to compute the stored strain energy $\mathcal{E}$ before and after crack onset, which is usually available in commercial FEM codes, and compute the variation of work of external loads during crack onset, which vanishes under displacement control. The disadvantage in the evaluation of $\Delta\mathcal{E} = \mathcal{E}_1 - \mathcal{E}_2$ may be that a highly accurate values of $\mathcal{E}$ are required because we are subtracting two, in general, large but similar values $\mathcal{E}_1 - \mathcal{E}_2$. Nevertheless, we have tested this procedure and it works using highly refined FEM meshes.

There are several proposals for the evaluation of energy dissipated due the crack onset, taking into account mode mixity [19, 16]. Nevertheless, this issue is currently under discussion.

Considering for the sake of simplicity of the following explanations a proportional loading governed by an applied nominal stress $\sigma_{\text{nom.}}$ and a characteristic material strength parameter σ_c, the above energetic condition can be rewritten in the following form

$$\frac{\sigma_{\text{nom.}}}{\sigma_c} \geq e(\mathbf{a}), \tag{6}$$

where the dimensionless function e defines the (hyper)surface (or curve if $m = 1$) of this incremental energy criterion defined for $\mathbf{a} \in A$. Recall that A is the considered feasible region in the parameter space. In a more complicated case, where an analytical or semi-analytical representation of e is not possible, this function is evaluated just at a suitably chosen finite number of parameter-vector values $\mathbf{a} \in A$ by a numerical method, e.g., using FEM, and then interpolated.

In many configurations studied by this coupled criterion assuming linear elastic behaviour of the material, this function has been computed explicitly, e.g., [17, 18, 20].

Coupled criterion. Dimensionless functions s and e are defined in the feasible parameter region A and we look for the minimum nominal load satisfying both criteria. Thus, we actually look for the minimum value in the intersection of epigraphs of these functions defined as,

$$\frac{\sigma_{\text{nominal},c}}{\sigma_c} = \min_{\mathbf{a}\in A} \max\left\{s(\mathbf{a}), e(\mathbf{a})\right\}. \tag{7}$$

This may lead to a special kind of nonsmooth optimization, which in simple configurations with typically $m = 1$ or 2, has been solved by many authors quite easily, but in general situation with the parameter space of a larger dimension $m \geq 3$ may require more sophisticated algorithms. Actually, this is a classical problem of optimization of functions with an envelope representation, which can be quite easily reformulated to a standard setting of minimization of a linear function subject to smooth nonlinear constraints, see [21] for details and further references.

When the new crack surface is parameterized by only one scalar parameter a, i.e. $m = 1$, the above non-smooth minimization problem (7) often leads to the solution of the nonlinear equation $s(a) = e(a)$. Nevertheless, sometimes depending on the structural parameters other scenarios are possible where the problem solution is governed by only one criterion, typically by the energetic one, e.g. for large values of the brittleness number corresponding to tough (ductile) configurations, in such cases we look for the minimum of a usually smooth function $e(a)$, cf. [17, 18, 20]. However, the formulation in (7) may sometimes lead to non-unique solutions for the crack length at onset, typically in cases where a new crack surface is placed in a uniform stress field before crack onset.

Finally, in order to better understand the whole fracture process predicted by the coupled criterion of FFM in the case of a finite-crack onset, and also its relations to the predictions of the LEFM, it is very illustrative to observe this process in usual load-displacement diagrams shown in Fig. 1, taken from [16], cf. [22]. In these plots, a dot-dashed line represents linear elastic equilibrium path without fracture, and a dashed curve represents the equilibrium paths predicted by the LEFM, this curve comes from infinity as LEFM predicts an infinite failure load in such a case (representing a kind of snap-back instability at infinity). A continuous piecewise defined line indicated by 0-1-2 represents the path predicted by the coupled criterion of FFM, in the case of load control in (a) and (c) plots, and in the case of displacement control in (b) and (d) plots. Plots (a) and (b) correspond to a problem where after the crack onset a stable crack propagation is not possible, whereas in plots (c) and (d), after some unstable crack growth, crack arrest is predicted, and then a stable crack growth can begin.

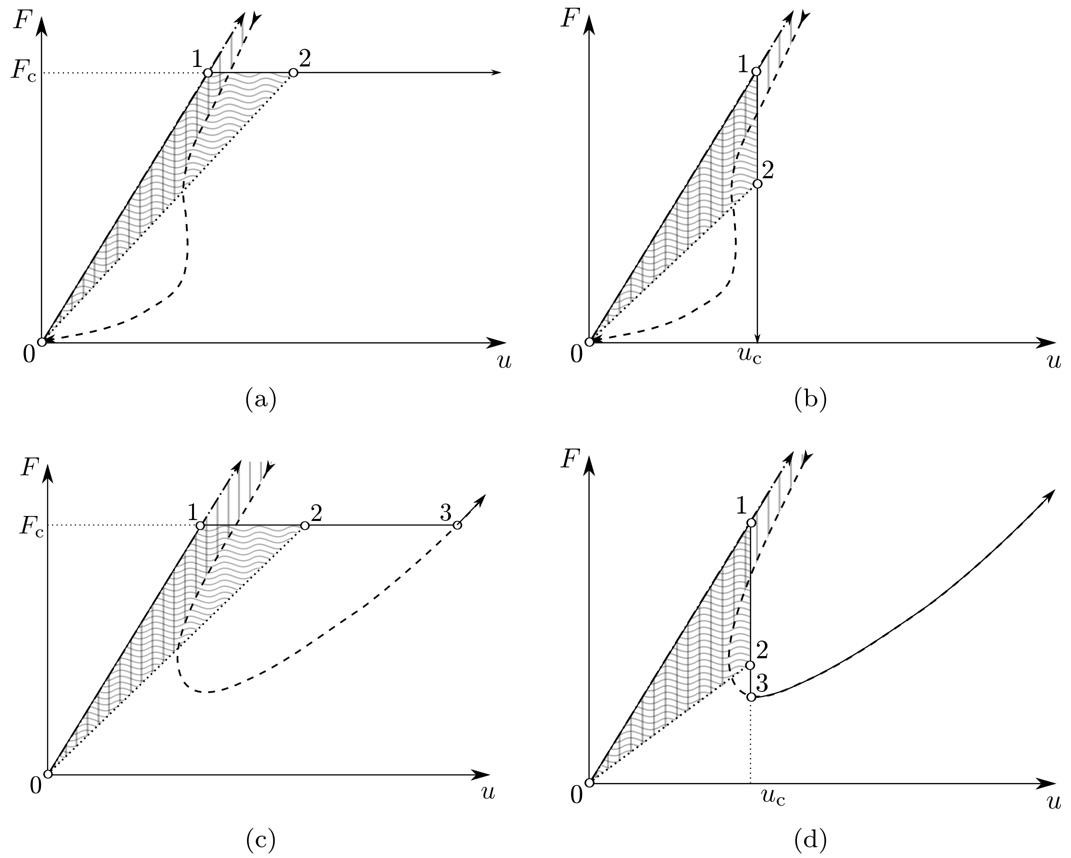

Figure 1. Examples of load-displacement curves predicted by the coupled criterion of FFM for cases with and without global failure immediately after the crack onset, respectively, (a)&(b) and (c)&(d), and load-controlled and displacement-controlled tests, respectively, (a)&(c) and (b)&(d). The main steps of the problem evolution predicted by the coupled criterion are denoted as: (0-1) linear-elastic loading, (1-2) crack onset, (2-3) unstable crack growth after the onset, (3-) subsequent stable growth (from García [16]).

The energetic criterion of FFM requires that the two areas shown in each plot of Fig. 1, representing the dissipated energy due to the crack formation, are of the same magnitude. These areas are: *i)* the triangle 0-1-2 in the case of FFM, and *ii)* that bounded by the linear elastic path, the LEFM equilibrium path up to achieving the crack length predicted by FFM and then the unloading straight line, in the hypothetical case of LEFM. Detailed explanations and an example of such diagrams in a particular case of debond onset at a fibre-matrix interface under transverse loads can be found in [16, 22].

3.2. Applications of the coupled stress and energy criterion of FFM to crack onset in composites

In last years the coupled criterion of FFM has been successfully applied to analyse several fracture and damage mechanisms in composites on different scales, providing new predictions about failure loads and size effect, see [12, 16] for a comprehensive review:

- Debond onset at fibre-matrix interface under transverse loads [17, 18, 19, 23, 22].
- Debond onset at spherical particle-matrix interface under transverse loads [24].
- Transverse crack onset in the 90° ply in a 0°/90° cross-ply subjected to a longitudinal loading [20].
- Crack onset in open hole composite laminates under tension [25, 26].

4. An insight into the coupled stress and energy criterion of FFM. A generalization of LEFM

4.1. Principle of minimum total potential energy subject to a stress condition (SC-PMTE)

The present proposal is to apply the principle of minimum total energy to the problem of crack onset and also crack propagation, assuming quasistatic problem evolution, i.e. inertial forces are neglected. Let us consider a given applied load (under load or displacement control), then the alternative formulation of the coupled criterion in general terms is

$$\begin{aligned}&\text{minimize } \ \Pi(\Delta S_{\mathrm{C}}) + \mathcal{R}(\Delta S_{\mathrm{C}})\\&\text{subject to stress condition,}\end{aligned} \tag{8}$$

where $\Pi(\Delta S_{\mathrm{C}})$ represents the potential energy for the given load and the considered new crack surface ΔS_{C}, and $\mathcal{R}(\Delta S_{\mathrm{C}})$ gives the energy dissipated up to the formation of this new crack surface. Under displacement control, this potential energy is given by the stored strain energy, i.e. $\Pi = \mathcal{E}$. In the case of brittle fracture, $\mathcal{R}$ is essentially given by Griffith's surface energy, i.e. $\mathcal{R}(\Delta S_{\mathrm{C}}) = \Gamma(\Delta S_{\mathrm{C}})$. Nevertheless, in general, $\mathcal{R}$ could additionally include energy dissipated due to other dissipative phenomena as friction, plasticity, viscosity, etc. The stress constraint used in (8) means that the total energy minimization considers only those new crack surface configurations for which the stress criterion is verified for the given load.

A mathematical formulation of this constraint minimization problem can be expressed as

$$\min_{\mathbf{a}\in A_\sigma} \ \Pi(\Delta S_{\mathrm{C}}(\mathbf{a})) + \mathcal{R}(\Delta S_{\mathrm{C}}(\mathbf{a})), \tag{9}$$

where the feasible region

$$A_\sigma = \left\{\mathbf{a} \in A \mid f\left(\sigma_{ij}(\cdot), \Delta S_c(\mathbf{a})\right) \geq \sigma_c\right\} \subset A \tag{10}$$

depends on the applied load value and gathers those new crack surface $\Delta S_c(\mathbf{a})$ configurations verifying the stress criterion. The solution of the above minimization problem will be denoted as $\mathbf{a}^*$. We will refer to the above defined approach as *Stress-Constrained Principle of Minimum Total Energy* (SC-PMTE). Defining the initial configuration by $\mathbf{a}_0$, $\Delta S_C(\mathbf{a}_0) = \emptyset$ represents the initial configuration, and a new crack surface may appear only if there is another configuration with the value of $\Pi + \mathcal{R}$ lower than or equal to that of the initial configuration. A_σ can be empty, $A_\sigma = \emptyset$, typically for small applied loads in the case of crack onset at a stress concentrator or in the case of uniform stresses along the potential crack surface. In the former case A_σ will change progressively with increasing load once a sufficiently high load is applied, whereas in the latter case it will jump from $A_\sigma = \emptyset$ to a region of a finite measure at a critical load. In the case of a stress singularity, e.g., an existing crack, A_σ can be non-empty even for small load values.

The idea behind this formulation is that a new crack surface can appear only in those regions where sufficiently high stresses are applied before fracture. Therefore, the crack formation is inhibited if stresses are too small, although there is a sufficient amount of energy available to be released, e.g., in a large bulk. This fact is related to the difficulties with the application of the PMTE without any stress constraint to the present problem of crack onset or propagation, because the PMTE looks for the energy only and any stress condition is missing, which sometimes leads to too early fracture predictions, theoretically for any small load if the bulk is sufficiently large and load control is considered. Several concepts of solutions essentially based on the PMTE, sometimes referred to as energetic solutions, have been studied in depth by mathematicians in the last two decades, see [7, 8].

Notice that a parameterization of $\Delta S_C(\mathbf{a})$ is not strictly required in the formulation (as we could just consider a set, in general infinite, of all cracks considered in the minimization procedure) and is used in the expressions presented in order to be more specific and provide a formulation directly implementable in a computational code.

The principle of minimum total energy in (9) leads to searching for a global minimum, or strictly speaking infimum, as in some situations there is no global minimum because $\Pi + \mathcal{R} \to -\infty$ for some cracked configurations. Nevertheless, in complex problems, where in addition to a global minimum there are some local minima with values of $\Pi + \mathcal{R}$ smaller than that of the initial configuration, maybe one should explore also these energetically advantageous local minima, specially those located closer to the initial configuration than the global minimum.

The minimization problem (9) can be rewritten in terms of changes of energies due to the new crack onset or growth with respect to the initial state, from which the crack onset or growth is considered in the minimization problem, because the constant values of energies associated to the initial state have no influence on the result of minimization,

$$\min_{\mathbf{a}\in A_\sigma} \Delta\Pi(\Delta S_C(\mathbf{a})) + \Delta\mathcal{R}(\Delta S_C(\mathbf{a})). \qquad (11)$$

In fact, this is a formulation of the (incremental) principle of maximum decrease of the total energy, which is equivalent to the above principle of minimum total energy, subjected to a stress condition. We could also refer to (11) as the principle of maximum excess of total energy at the crack onset. Usually, when $\Delta\Pi + \Delta\mathcal{R} < 0$, this excess of the energy released due to crack onset will mainly go to a kinetic energy increase.

If the energy excess is relatively high, dynamic effects may become relevant and the present assumption of quasistatic solution evolution (including even finite cruck jumps, advances), neglecting inertial effects, may not further represent a sufficiently accurate approximation of the real behaviour and a dynamic fracture analysis may be required.

It should be mentioned that a related principle was proposed by Lawn [27] looking for the maximum decrease of the total energy with respect to the area of a new crack, but without any assumption on stresses prior to fracture, see also [28, 29]. Note that, Lawn's proposal corresponds to the differential condition of the maximum energy release rate in classical Fracture Mechanics.

For the solution of (11) obviously holds

$$\Delta\Pi(\Delta S_C(\mathbf{a}^*)) + \Delta\mathcal{R}(\Delta S_C(\mathbf{a}^*)) \leq 0, \quad \text{and then} \quad -\Delta\Pi(\Delta S_C(\mathbf{a}^*)) \geq \Delta\mathcal{R}(\Delta S_C(\mathbf{a}^*)), \tag{12}$$

because $\Delta\Pi + \Delta\mathcal{R} = 0$ for the initial state, with $\Delta S_C(\mathbf{a}^*) = \emptyset$. Recall that $\Delta\mathcal{R} > 0$. The inequalities (12) represent in fact the energetic condition (5) of the original formulation of the coupled criterion (7). However, the principle of minimum total energy provides a more selective criterion looking for a minimum of $\Delta\Pi + \Delta\mathcal{R}$), not merely for solutions satisfying (12) as in the case of the original formulation of the coupled criterion. Hence, the new formulation is able to predict new crack area even in those cases where the prediction by the original formulation of the coupled criterion is non-unique. Nevertheless, the critical load predicted by both formulations is identical.

The fact that we are looking for a (global) minimum of a function in a region leads naturally to a finite crack advance $\Delta S_C(\mathbf{a}^*)$, which may imply that the crack jump is actually associated to a tunneling effect through a total energy barrier as is schematically shown in Fig. 2 for different sizes of the feasible region A_σ.

4.2. Relation of SC-PMTE to Griffith's formulation of LEFM

Griffith's formulation of LEFM has been shown to be very successful in prediction of classical crack propagation using only the concept of fracture toughness or fracture energy, and need not the material strength concept. Thus, we can ask: Why we need the coupled stress and energy criterion of FFM, or the above introduced SC-PMTE, to characterize fracture of materials in general, or in other terms why we should incorporate material strength into a general fracture criterion? An explanation why LEFM is able to make correct predictions without material strength is that the applications of LEFM are essentially restricted to predict a continuous advance of a priori existing classical cracks (and, as will be explained below, also to crack formation at strong singularities). In this case, the following two conditions are fulfilled:

- continuous crack growth by infinitesimal advances, and
- infinite stresses ahead of the crack tip.

Hence, any stress criterion would be fulfilled for these infinitesimal crack advances, which explains why the original Griffith's theory does not need any stress criterion. In all other situations,

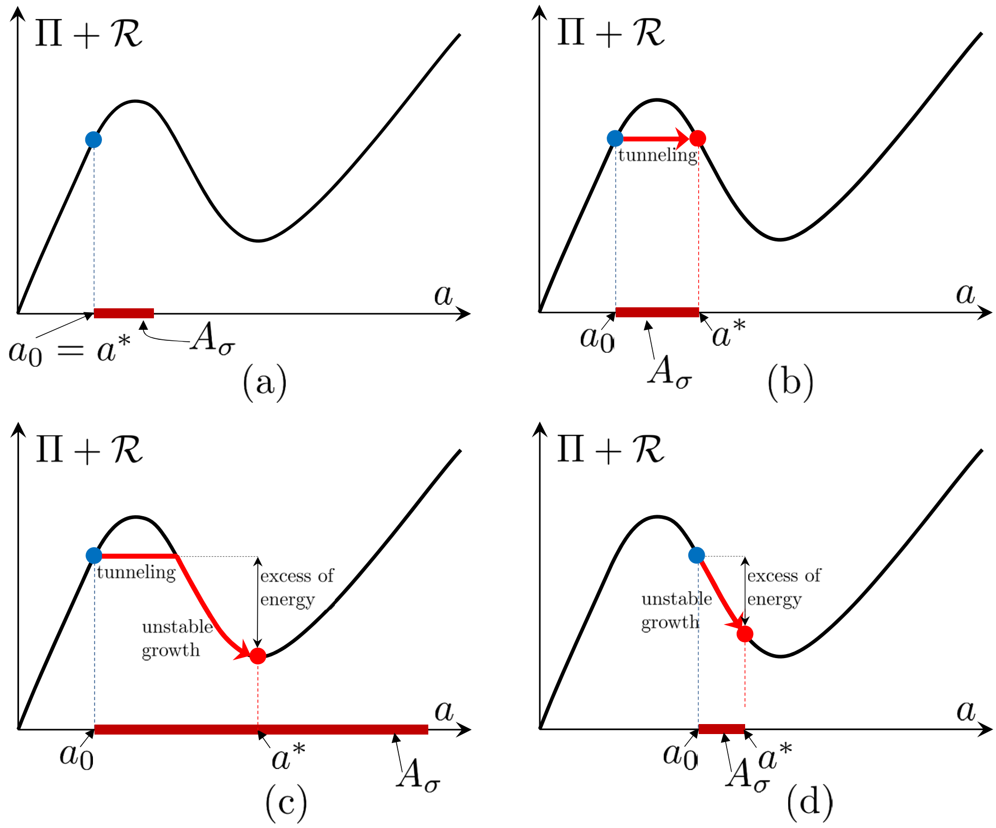

Figure 2. Examples of the application of the principle of minimum total energy subject to the stress-criterion constraint. (a) Stresses at the potential crack surface, which would cross the total energy barrier, are too small, hence the stress condition inhibits a crack onset, and consequently $\mathbf{a}_0 = \mathbf{a}^*$, (b) Stresses at the potential crack surface are sufficiently high, thus the stress condition allows a crack onset by tunneling through the total energy barrier, $\Delta\Pi(\Delta S_C(\mathbf{a}^*)) + \Delta\mathcal{R}(\Delta S_C(\mathbf{a}^*)) = 0$, (c) Stresses are sufficiently high in a large region including the potential crack surface (situation typical for fracture in a region of uniform stresses), thus the stress condition allows a crack onset by tunneling through the total energy barrier and subsequent unstable crack growth, $\Delta\Pi(\Delta S_C(\mathbf{a}^*)) + \Delta\mathcal{R}(\Delta S_C(\mathbf{a}^*)) < 0$, (d) Stresses are sufficiently high only in a small region, not covering the complete decreasing part of the potential energy hill, $\Delta\Pi(\Delta S_C(\mathbf{a}^*)) + \Delta\mathcal{R}(\Delta S_C(\mathbf{a}^*)) < 0$.

where such continuous crack growth cannot be predicted, and, thus, where we consider discontinuous stepwise crack advance breaking material in zones away from the crack tip, we need to incorporate a strength parameter into a stress condition to guarantee sufficiently high stresses in regions where fracture will subsequently and suddenly happen. Thus, a stress condition should be incorporated in a general fracture criterion to avoid unphysical material breakage in zones subjected to too low stresses.

In the limit $\sigma_c \to \infty$, the predictions by SC-PMTE converge to predictions by LEFM (Griffith theory), i.e. being coincident for "classical cracks" (referring to cracks with stress singularity $\sigma_{ij} \sim r^{-0.5}$ at the crack tip), but no fracture is predicted at weak singularities (referring to points with stress singularity $\sigma_{ij} \sim r^{\lambda-1}$ with $0.5 < \lambda < 1$), stress concentrations and regions with uniform stresses, under quasistatic loading.

In this sense the present formulation of SC-PMTE can be understood as a generalization of LEFM relaxing the too restrictive condition $\sigma_c \to \infty$ in order to be able to assess crack onset at weak singularities, stress concentrations, etc. This relaxation, nevertheless, has a consequence, namely slightly different failure load predicted for initiation of unstable crack growth and slightly different values of fracture energy obtained from fracture tests with unstable crack growth. Let us explain these differences in detail.

Let the fracture energy of a material be determined by experiments with stable growth of existing cracks, assuming a simple standard situation with just one local minimum coincident with the global minimum of $\Pi + \mathcal{R}$. In this case, both LEFM and SC-PMTE procedures will determine the same value of fracture energy denoted as G_c, specifically

$$\sigma^{\text{SC-PMTE}}_{\text{nom.,stable}}(G_c) = \sigma^{\text{LEFM}}_{\text{nom.,stable}}(G_c). \tag{13}$$

The reason for this accordance is that both procedures predict infinitesimal crack advances and the stress criterion in SC-PMTE does not play any role because of infinite stresses ahead of the crack tip. If this value of G_c is used for prediction of failure load originating unstable growth of an existing crack, then there will be a slight difference between these two predictions, specifically

$$\sigma^{\text{SC-PMTE}}_{\text{nom.,unstable}}(G_c) < \sigma^{\text{LEFM}}_{\text{nom.,unstable}}(G_c). \tag{14}$$

Cornetti et al. [10] presented the values verifying (14) for a Griffith crack of length $2a$ in Mode I in an infinite isotropic plate as functions of initial crack length using the average tensile stress criterion $(1)_2$, whereas Figure 3 shows these values computed using the pointwise tensile stress criterion in SC-PMTE, with $r_I = \frac{1}{\pi}\frac{G_c E}{\sigma_c^2}$ denoting Irwin's characteristic length.

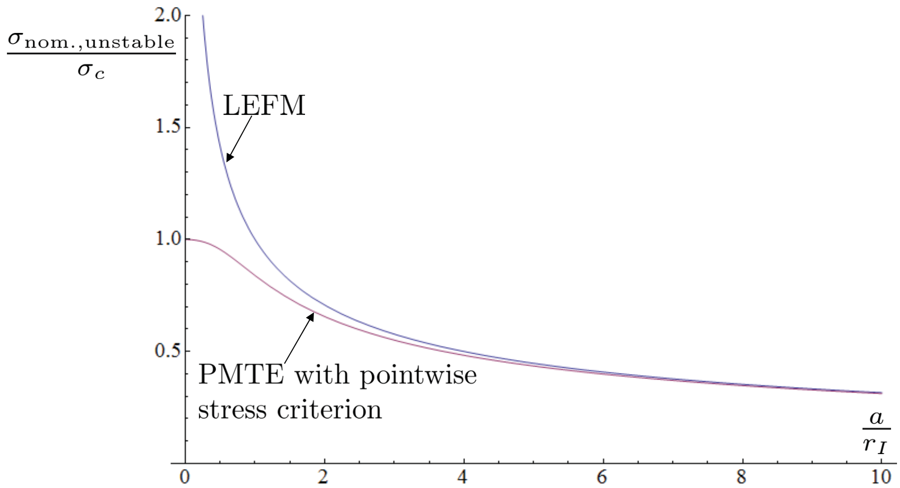

Figure 3. Failure nominal stresses for a Griffith crack in Mode I in an infinite plate predicted by LEFM and SC-PMTE using pointwise stress criterion.

If viceversa, experiments with unstable growth of existing cracks are used to measure fracture energy, then we obtain different fracture energies by these two procedures (indicated by a superscript), i.e.

$$\sigma^{\text{SC-PMTE}}_{\text{nom.,unstable}}(G_c^{\text{SC-PMTE}}) = \sigma^{\text{LEFM}}_{\text{nom.,unstable}}(G_c^{\text{LEFM}}) \quad \Rightarrow \quad G_c^{\text{SC-PMTE}} > G_c^{\text{LEFM}}, \tag{15}$$

where the last inequality is obtained in view of (14). An example of the measured values of fracture toughness for composite laminates verifying the inequality in (15) can be found in [25]. In fact, the value $G_c^{\text{SC-PMTE}}$ obtained by evaluating an experiment applying the SC-PMTE would be more coherent and should be applied in subsequent fracture studies of such a material by the SC-PMTE.

Although the above explained differences may be expected to be quite small, one can try to measure/detect them by careful and accurate discriminating experiments in future, which would either support or oppose the present proposal of SC-PMTE as a generalization of LEFM. To increase these differences, one could try, in view of Figure 3, to test specimens including small cracks and made from brittle materials of small strength σ_c where initiation of unstable crack growth is originated.

It seems very interesting to review the relation between SC-PMTE and LEFM predictions for the loads originating a crack propagation or onset in fracture problems with different degree of severity of the linear elastic stress state[1]. Simple standard configurations with proportional and increasing applied loads or displacements are considered, and the same fracture properties are assumed by both procedures.

Strong singularity. In this case the stress state in the neighbourhood of a singular point is more severe than in the case of a classical crack. An example of such a situation in composites is the neighborhood of crack front in a fibre broken under tension, see [30]. Specifically, in the neighborhood of a strong singularity point the behaviour of the singular asymptotic term of stress solution in 2D is described by $\sigma_{ij}(r, \theta) \sim r^{\lambda-1}$, where r is the distance to the singular point and the singularity exponent $0 < \lambda < 0.5$. At a strong singularity, both procedures, LEFM and SC-PMTE, predict essentially identical fracture behaviour, independent of the material strength value considered in SC-PMTE, at least from the very beginning of fracture process: a crack appears at the singular point for any nonzero load, its length a is increasing continuously with increasing nominal load following the law $a \sim \sigma_{\text{nom.}}^{\frac{2}{1-2\lambda}}$. The ERR of this crack is $G(a) \sim a^{2\lambda-1}$, thus $G(a) \to \infty$ for $a \to 0$. This behaviour was studied in detail in the analysis of the fragmentation test in [30]. The reason for the coincident predictions of fracture behaviour by both procedures is that crack advances are infinitesimal with infinite stresses ahead of the crack tip, thus the stress criterion plays no role here.

Existing classical crack. This case has been discussed in detail above. A general conclusion is that if a stable smooth crack growth (by infinitesimal advances with increasing loading) is predicted by SC-PMTE then the failure load predictions will coincide. However, if an unstable crack growth initiated by a crack jump is predicted by SC-PMTE, the failure load predicted will be somewhat lower than that predicted by LEFM, see Figure 3.

Weak singularity. In this case the stress state in the neighbourhood of a singular point is less severe than in the case of a classical crack. An example of such a situation in composites is the neighborhood of multimaterial corners in adhesively bonded lap joints [31], and also the neighborhood of transverse crack tip in 90° ply terminating at the interface with

[1]For the sake of simplicity of explanations and expressions, we will not consider singular oscillatory behaviour of stresses described using complex singularity exponents λ.

0° ply in a 0°/90° cross ply [32]. At a weak singularity point the singular asymptotic term of stress solution in 2D is described by $\sigma_{ij}(r,\theta) \sim r^{\lambda-1}$, where the singularity exponent $0.5 < \lambda < 1$. The ERR of a crack of length a growing from the singular point is $G(a) \sim a^{2\lambda-1}$, thus $G(a) \to 0$ for $a \to 0$. Hence, at a weak singularity, LEFM predicts an infinite failure load, whereas SC-PMTE is able to predict finite failure loads originating an abrupt onset of a small crack of a finite length which reasonably agrees with experiments. It is easy to see that this sudden crack formation is associated to tunneling an energetic barrier of $\Pi + \mathcal{R}$.

No stress singularity - bounded stresses. Stress concentration points and regions of uniform stresses belong to this case. Debond onset at fibre-matrix interface is an example of the former case [33, 17, 18], and transverse crack onset in 90° ply of a 0°/90° cross ply [32, 20] of the latter case. The ERR of a crack of length a is $G(a) \sim a$, thus $G(a) \to 0$ for $a \to 0$. Hence, the situation is similar to the above weak singularity case, LEFM predicts an infinite failure load, whereas SC-PMTE predicts finite failure loads originating an abrupt onset of a small crack of a finite length in agreement with experiments. This crack onset is also here associated to tunneling an energetic barrier of $\Pi + \mathcal{R}$.

As can be deduced form the above brief review, differences between LEFM and SC-PMTE increase with decreasing severity of stress state, which can be interpreted in the present framework as a consequence of assuming an infinite strength $\sigma_c \to \infty$ in Griffith's criterion of LEFM.

It should be mentioned that PMTE without a stress criterion constraint, used widely in mathematically oriented works on fracture, may sometimes predict identical failure load as SC-PMTE, nevertheless in configurations under load control and large (theoretically infinite) bulk it leads to vanishing failure load predictions, which in general disagrees with experiments.

5. Conclusions

The coupled criterion of FFM is a pragmatic and efficient approach to characterize sudden damage initiation and propagation in form of cracks in virtually all structural configurations, covering several types of stress singularities, stress concentrations and uniform stress fields. It also suitable for non-classical configurations of cracks with different singularities at the crack tip, as is the case of a crack approaching an interface, where the crack can stop, deflect or cross.

The present work proposes a new insight into the coupled stress and energy criterion of FFM. A new alternative and general formulation of the coupled criterion as a global minimization of the total energy under a stress criterion constraint shows that a finite crack jump may be associated to a tunneling effect across the total energy barrier, by breaking material bonds across a surface of a finite area in the material subjected to sufficiently high stresses prior to fracture. It is expected that this new formulation will allow implementations of general and efficient computational procedures in future.

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