FREE VIBRATION OF AXIALLY LOADED FUNCTIONALLY GRADED SANDWICH BEAMS USING REFINED SHEAR DEFORMATION THEORY

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Abstract

Free vibration of axially loaded functionally graded (FG) sandwich beams using refined shear deformation theory is presented. Sandwich beams with FG skins-homogeneous core and homogeneous skins-FG core are considered. Governing equations of motion are derived from the Hamilton's principle. Finite element model (FEM) and Navier solution are developed to determine the natural frequencies and critical buckling loads of FG sandwich beam for various power-law index, skin-core-skin thickness ratios and boundary conditions.

1. Introduction

In recent years, due to high strength-to-weight ratio, there is a rapid increase in the use of FG sandwich structures in aerospace, marine and civil engineering. With the wide application of sandwich structures, understanding their vibration response becomes an important task. However, this complicated problem has not well-researched and only few related documented investigation exists. Amirani et al. [1] used the element free Galerkin method to study free vibration analysis of sandwich beam with FG core. Bui et al. [2] investigated transient responses and natural frequencies of sandwich beams with inhomogeneous FG core using a truly meshfree radial point interpolation method. Mashat et al. [3] used Carrera Unified Formulation to perform vibration of sandwich beam with FG core. As far as authors are aware, there is no work available using refined shear deformation theory to study vibration of axially loaded FG sandwich beams. In this paper, which is an extension of previous work [4], free vibration of axially loaded FG sandwich beams using refined shear deformation theory is presented. Sandwich beams with FG skins-homogeneous core and homogeneous skins-FG core are considered. Governing equations of motion are derived from the Hamilton's principle. FEM and Navier solution are developed to determine the natural frequencies and critical buckling loads of FG sandwich beam for various power-law index, skin-core-skin thickness ratios and boundary conditions.

2. FG sandwich beams

Consider a FG sandwich beam with length l and rectangular cross-section $b\times h$, with b being the width and h being the height. The x-, y-, and z-axes are taken along the length, width, and height

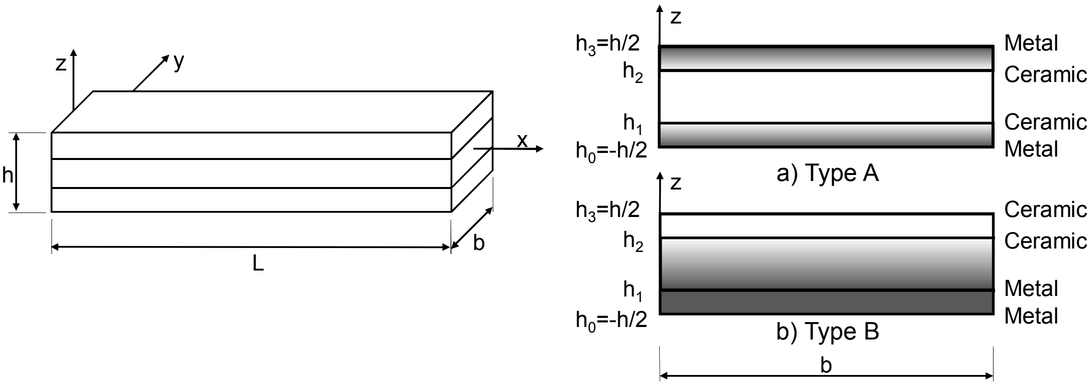

Figure 1: Geometry and coordinate of a FG sandwich beam.

of the beam, respectively, as illustrated in Fig. 1. For the brevity, the ratio of the thickness of each layer from bottom to top is denoted by the combination of three numbers, i.e. 1-0-1, 1-1-1, and so on. Young's modulus E and mass density ρ are assumed to vary continuously through the beam depth by a power-law distribution [5] as :

$$E(z) = (E_c - E_m)V_c + E_m \tag{1a}$$
$$\rho(z) = (\rho_c - \rho_m)V_c + \rho_m \tag{1b}$$

where subscripts m and c represent the metallic and ceramic constituents, V_c is volume fraction of the ceramic phase of the beam, respectively. Two different types of FG beam are studied:

2.1. Type A: sandwich beam with FG skins and homogeneous core

The core is fully ceramic and skins are composed of a FG material. The bottom skin varies from a metal-rich surface ($z = -h/2$) to a ceramic-rich surface while the top skin face varies from a ceramic-rich surface to a metal-rich surface ($z = h/2$) (Fig. 1a). The volume fraction of the ceramic phase is obtained as:

$$\begin{cases} V_c = \left(\frac{z-h_o}{h_1-h_0}\right)^p, & z \in [-h/2, h_1] \quad \text{(bottom skin)} \\ V_c = 1, & z \in [h_1, h_2] \quad \text{(core)} \\ V_c = \left(\frac{z-h_3}{h_2-h_3}\right)^p, & z \in [h_2, h/2] \quad \text{(top skin)} \end{cases} \tag{2}$$

where p is the power-law index.

2.2. Type B: sandwich beam with homogeneous skins and FG core

The bottom skin is fully metal and the top skin is fully ceramic, while, the core layer is graded from metal to ceramic (Fig. 1b). The volume fraction of the ceramic phase in the core is obtained by:

$$V_c = (\frac{z - h_1}{h_2 - h_1})^p, \quad z \in [h_1, h_2] \tag{3}$$

3. Variational Formulation

The displacement field of the present theory can be obtained as:

$$U(x,z,t) = u(x,t) - zw_b'(x,t) - \frac{4z^3}{3h^2}w_s'(x,t) = u(x,t) - zw_b'(x,t) - f(z)w_s'(x,t) \quad (4a)$$
$$W(x,z,t) = w_b(x,t) + w_s(x,t) \quad (4b)$$

where u is the axial displacement, w_b and w_s are the bending and shear components of transverse displacement along the mid-plane of the beam. The non-zero strains are given by:

$$\epsilon_x = \frac{\partial U}{\partial x} = u' - zw_b'' - fw_s'' = \epsilon_x^\circ + z\kappa_x^b + f\kappa_x^s \quad (5a)$$
$$\gamma_{xz} = \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} = (1 - \frac{df}{dz})w_s' = g\gamma_{xz}^\circ \quad (5b)$$

The variation of the strain energy can be stated as:

$$\delta\mathcal{U} = \int_0^v (\sigma_x\delta\epsilon_x + \sigma_{xz}\delta\gamma_{xz})dv = \int_0^l (N_x\delta\epsilon_z^\circ + M_x^b\delta\kappa_x^b + M_x^s\delta\kappa_x^s + Q_{xz}\delta\gamma_{xz}^\circ)dx \quad (6)$$

where N_x, M_x^b, M_x^s and Q_{xz} are the stress resultants, defined as:

$$(N_x, M_x^b, M_x^s) = \int_{-h/2}^{h/2} \sigma_x(1, z, f)bdz \quad (7a)$$
$$Q_{xz} = \int_{-h/2}^{h/2} \sigma_{xz}gbdz \quad (7b)$$

The variation of the potential energy by the axial force P_0 can be written as:

$$\delta\mathcal{V} = -\int_0^l P_0\Big[\delta w_b'(w_b' + w_s') + \delta w_s'(w_b' + w_s')\Big]dx \quad (8)$$

The variation of the kinetic energy can be expressed as:

$$\delta\mathcal{K} = \int_0^v \rho(\dot{U}\delta\dot{U} + \dot{W}\delta\dot{W})dv = \int_0^l \Big[\delta\dot{u}(m_0\dot{u} - m_1\dot{w}_b{}' - m_f\dot{w}_s{}') + \delta\dot{w}_b m_0(\dot{w}_b + \dot{w}_s) + \delta\dot{w}_s m_0(\dot{w}_b + \dot{w}_s)$$
$$+ \delta\dot{w}_b{}'(-m_1\dot{u} + m_2\dot{w}_b{}' + m_{fz}\dot{w}_s{}') + \delta\dot{w}_s{}'(-m_f\dot{u} + m_{fz}\dot{w}_b{}' + m_{f^2}\dot{w}_s{}')\Big]dx \quad (9)$$

where:

$$(m_0, m_1, m_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2)bdz \quad (10a)$$
$$(m_f, m_{fz}, m_{f^2}) = \int_{-h/2}^{h/2} \rho(f, fz, f^2)bdz \quad (10b)$$

In order to derive the equations of motion, Hamilton's principle is used:

$$0 = \int_{t_1}^{t_2} (\delta\mathcal{K} - \delta\mathcal{U} - \delta\mathcal{V})dt = \int_{t_1}^{t_2}\int_0^l \Big[\delta\dot{u}(m_0\dot{u} - m_1\dot{w}_b{}' - m_f\dot{w}_s{}') + \delta\dot{w}_b m_0(\dot{w}_b + \dot{w}_s)$$
$$+ \delta\dot{w}_b{}'(-m_1\dot{u} + m_2\dot{w}_b{}' + m_{fz}\dot{w}_s{}') + \delta\dot{w}_s m_0(\dot{w}_b + \dot{w}_s) + \delta\dot{w}_s{}'(-m_f\dot{u} + m_{fz}\dot{w}_b{}' + m_{f^2}\dot{w}_s{}')$$
$$+ P_0[\delta w_b'(w_b' + w_s') + \delta w_s'(w_b' + w_s')] - N_x\delta u' + M_x^b\delta w_b'' + M_x^s\delta w_s'' - Q_{xz}\delta w_s'\Big]dxdt \quad (11)$$

The constitutive relations of a FG sandwich beam can be written as:

$$\sigma_x = E\epsilon_x \tag{12a}$$
$$\sigma_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} = G\gamma_{xz} \tag{12b}$$

By using Eqs. (5), (7) and (12), the constitutive equations for stress resultants and strains are obtained:

$$\begin{Bmatrix} N_x \\ M_x^b \\ M_x^s \\ Q_{xz} \end{Bmatrix} = \begin{bmatrix} A & B & B_s & 0 \\ & D & D_s & 0 \\ & & H_s & 0 \\ \text{sym.} & & & A_s \end{bmatrix} \begin{Bmatrix} \epsilon_x^\circ \\ \kappa_x^b \\ \kappa_x^s \\ \gamma_{xz}^\circ \end{Bmatrix} \tag{13}$$

where:

$$(A, B, B_s, D, D_s, H_s) = \int_{-h/2}^{h/2} E(1, z, f, z^2, fz, f^2)bdz \tag{14a}$$
$$A_s = \int_{-h/2}^{h/2} g^2 Gbdz \tag{14b}$$

4. Governing Equations of Motion

The equilibrium equations of the present study can be obtained by integrating the derivatives of the varied quantities by parts and collecting the coefficients of δu, δw_b and δw_s:

$$N_x' = m_0\ddot{u} - m_1\ddot{w}_b{}' - m_f\ddot{w}_s{}' \tag{15a}$$
$$M_x^{b\prime\prime} - P_0(w_b'' + w_s'') = m_0(\ddot{w}_b + \ddot{w}_s) + m_1\ddot{u}' - m_2\ddot{w}_b{}'' - m_{fz}\ddot{w}_s{}'' \tag{15b}$$
$$M_x^{s\prime\prime} + Q_{xz}' - P_0(w_b'' + w_s'') = m_0(\ddot{w}_b + \ddot{w}_s) + m_f\ddot{u}' - m_{fz}\ddot{w}_b{}'' - m_{f^2}\ddot{w}_s{}'' \tag{15c}$$

By substituting Eq. (13) into Eq. (15), the explicit form of the governing equations of motion can be expressed:

$$Au'' - Bw_b''' - B_S w_s''' = m_0\ddot{u} - m_1\ddot{w}_b{}' - m_f\ddot{w}_s{}' \tag{16a}$$
$$\begin{aligned} Bu''' - Dw_b^{iv} - D_s w_s^{iv} - P_0(w_b'' + w_s'') &= m_0(\ddot{w}_b + \ddot{w}_s) + m_1\ddot{u}' \\ &- m_2\ddot{w}_b{}'' - m_{fz}\ddot{w}_s{}'' \end{aligned} \tag{16b}$$
$$\begin{aligned} B_s u''' - D_s w_b^{iv} - H_s w_s^{iv} + A_s w_s'' - P_0(w_b'' + w_s'') &= m_0(\ddot{w}_b + \ddot{w}_s) + m_f\ddot{u}' \\ &- m_{fz}\ddot{w}_b{}'' - m_{f^2}\ddot{w}_s{}'' \end{aligned} \tag{16c}$$

5. Analytical solution for simply-supported FG sandwich beams

The Navier solution procedure is used to obtain the analytical solutions for the simply-supported boundary conditions. The solution is assumed to be of the form:

$$u(x,t) = \sum_n^\infty U_n \cos\alpha x, e^{i\omega t} \tag{17a}$$

$$w_b(x,t) = \sum_n^{\infty} W_{bn} \cos \alpha x, e^{i\omega t} \quad (17b)$$

$$w_s(x,t) = \sum_n^{\infty} W_{sn} \sin \alpha x, e^{i\omega t} \quad (17c)$$

where ω is the natural frequency, $\sqrt{i}$=-1 the imaginary unit, $\alpha = n\pi/L$ and U^n={U_n, W_{bn}, W_{sn}}^{T} are the unknown maximum displacement coefficients. Substituting the Eq. (17) into Eq. (16), the analytical solutions can be obtained from the following equations:

$$([K^n] - P_0[G^n] - \omega^2[M^n])\{U^n\} = \{0\} \quad (18)$$

where the mass matrix $[M^n]$ and the stiffness matrix $[K^n]$ are given in [6], while the $[G^n]$ is given as follows:

$$G^n_{ij} = 0 \text{ except } G^n_{22} = G^n_{23} = G^n_{33} = \alpha^2 \quad (19)$$

6. Finite Element Formulation

The generalized displacements are expressed over each element as a combination of the linear interpolation function Ψ_j for u and Hermite-cubic interpolation function ψ_j for w_b and w_s associated with node j and the nodal values:

$$u = \sum_{j=1}^{2} u_j \Psi_j \quad (20a)$$

$$w_b = \sum_{j=1}^{4} w_{bj} \psi_j \quad (20b)$$

$$w_s = \sum_{j=1}^{4} w_{sj} \psi_j \quad (20c)$$

Substituting these expressions in Eq. (20) into the corresponding weak statement in Eq. (11), the FEM of a typical element can be expressed as the standard eigenvalue problem:

$$([K] - P_0[G] - \omega^2[M])\{\Delta\} = \{0\} \quad (21)$$

where $[K]$, $[G]$ and $[M]$ are the element stiffness matrix, element geometric stiffness matrix and element mass matrix, which are given in [4] and $\{\Delta\} = \{u\ w_b\ w_s\}^{\mathrm{T}}$, respectively.

7. Numerical Examples

In this section, sandwich FG beams with span-height ratio ($L/h = 10$) made of Alumina and Aluminium with two different skin-core-skin thickness ratios (1-1-1 and 2-2-1) for both Type A and Type B are considered. FG material properties are assumed to be: Aluminum (Al: $E_m = 70$GPa, $\nu_m = 0.3, \rho_m = 2702$kg/m^3) and Alumina ($Al_2O_3$: $E_c = 380$GPa, $\nu_c = 0.3, \rho_c = 3960$kg/m^3). The following non-dimensional natural frequencies and critical buckling loads are used in this study:

$$\overline{\omega} = \frac{\omega L^2}{h} \sqrt{\frac{\rho_m}{E_m}} \quad (22a)$$

$$\overline{P}_{cr} = P_{cr} \frac{12L^2}{E_m h^3} \quad (22b)$$

The critical buckling loads and the first four natural frequencies of simply-supported beams by FEM and Navier solutions are compared in Tables 1-3. It is clear that the fundamental frequencies and critical buckling loads by FEM of Type A agree completely with those of previous paper [4]. Generally, it is observed that the FEM and Navier solutions are in good agreement, thus accuracy of the present model is established. As the power-law index increases, the natural frequencies and critical buckling loads decrease. However, the decrease of Type A is bigger than that of Type B.

Type	Method	p						
		0	0.2	0.5	1	2	5	10
Type A								
1-1-1	FEM	52.2378	42.5584	33.6202	25.6662	19.0255	14.1239	12.5953
	Navier	52.2378	42.5584	33.6202	25.6662	19.0255	14.1239	12.5953
2-2-1	FEM	52.2378	44.2873	36.7759	29.8556	23.7391	18.7226	16.9067
	Navier	52.2378	44.2873	36.7759	29.8556	23.7391	18.7226	16.9067
Type B								
1-1-1	FEM	24.9154	23.5163	22.1588	20.9094	19.8662	19.1561	18.9739
	Navier	24.9125	23.5132	22.1556	20.9060	19.8627	19.1526	18.9704
2-2-1	FEM	29.2611	26.6453	24.1244	21.8548	20.0655	19.0186	18.8153
	Navier	29.2590	26.6428	24.1214	21.8515	20.0621	19.0152	18.8120

Table 1: The critical buckling loads of simply-supported FG sandwich beams.

Type A	Mode	Method	p						
			0	0.2	0.5	1	2	5	10
1-1-1	1	FEM	5.3933	4.9569	4.4892	3.9994	3.5136	3.0916	2.9483
		Navier	5.3933	4.9569	4.4892	3.9994	3.5136	3.0916	2.9483
	2	FEM	20.6110	19.0353	17.3213	15.5021	13.6758	12.0723	11.5232
		Navier	20.6110	19.0353	17.3212	15.5021	13.6758	12.0723	11.5232
	3	FEM	30.2333	29.3532	28.3759	27.2821	26.0467	24.6353	23.9232
	4	FEM	43.4309	40.3596	36.9547	33.2767	29.5243	26.1818	25.0233
		Navier	43.4308	40.3595	36.9547	33.2767	29.5243	26.1817	25.0233
2-2-1	1	FEM	5.3933	5.0336	4.6511	4.2509	3.8465	3.4679	3.3186
		Navier	5.3933	5.0336	4.6511	4.2509	3.8465	3.4679	3.3186
	2	FEM	20.6110	19.3148	17.9192	16.4421	14.9337	13.5082	12.9425
		Navier	20.6110	19.3148	17.9192	16.4421	14.9337	13.5082	12.9425
	3	FEM	30.2333	29.5817	28.8778	28.1142	27.2821	26.3705	25.9262
	4	FEM	43.4309	40.9112	38.1546	35.1929	32.1264	29.1929	28.0188
		Navier	43.4308	40.9111	38.1545	35.1929	32.1264	29.1928	28.0187

Table 2: The first four natural frequencies of simply-supported FG sandwich beams (Type A).

Finally, effects of the axial force on the natural frequencies are investigated. The first three natural frequencies of (1-1-1) sandwich beams of Type A with and without axial force are given in Table 4. It can be seen that they decrease as the axial force changes from tension to compression. This holds irrespective of the consideration of boundary conditions. In order to investigate the effect of axial force on the natural frequencies further, the first three load-frequency curves for (1-1-1) sandwich beams with the power-law index $p = 1$ of Type B are plotted in Fig. 2. The characteristic of these curves is that the value of the axial force for which the natural frequency vanishes constitutes the critical buckling load. From this figure,

Type B	Mode	Method	p						
			0	0.2	0.5	1	2	5	10
1-1-1	1	FEM	3.9326	3.8570	3.7802	3.7081	3.6505	3.6214	3.6214
		Navier	3.9383	3.8639	3.7884	3.7177	3.6616	3.6339	3.6344
	2	FEM	15.1473	14.8226	14.4874	14.1622	13.8812	13.6928	13.6511
		Navier	15.2590	14.9567	14.6465	14.3486	14.0963	13.9354	13.9044
	3	FEM	26.5169	25.7951	25.0327	24.2403	23.4391	22.6608	22.3234
	4	FEM	33.2459	32.6308	31.9851	31.3353	30.7251	30.2142	30.0303
		Navier	32.7414	32.0540	31.3392	30.6336	29.9962	29.5061	29.3522
2-2-1	1	FEM	4.2034	4.0678	3.9264	3.7922	3.6886	3.6479	3.6555
		Navier	4.2071	4.0729	3.9332	3.8011	3.6997	3.6608	3.6689
	2	FEM	16.2034	15.6463	15.0566	14.4795	13.9961	13.7137	13.6704
		Navier	16.2759	15.7476	15.1921	14.6541	14.2125	13.9678	13.9355
	3	FEM	27.6974	26.7032	25.6065	24.4224	23.1967	21.9962	21.4683
	4	FEM	35.1294	34.0760	32.9573	31.8326	30.8022	29.9810	29.6881
		Navier	34.8510	33.6946	32.4637	31.2388	30.1593	29.3864	29.1534

Table 3: The first four natural frequencies of simply-supported FG sandwich beams (Type B).

the first, second and third buckling occur at $P = 20.909$, 79.769 and 166.661, respectively for simply-supported beam.

Boundary conditions	p	P_{cr}	$P_0 = -0.5P_{cr}$ (tension)			$P_0 = 0$ (no axial force)			$P_0 = 0.5P_{cr}$ (compression)		
			ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
Clamped-	0	194.3840	14.1501	34.1005	58.6617	11.6661	30.4141	54.2732	8.3516	26.1519	49.4829
Clamped	0.2	159.8940	13.0876	31.6716	54.8817	10.7868	28.2765	50.8754	7.7184	24.3581	46.5127
	0.5	127.5140	11.9271	28.9855	50.6143	9.8271	25.9041	47.0112	7.0282	22.3545	43.0965
	1	98.2383	10.6911	26.0903	45.9219	8.8058	23.3391	42.7337	6.2946	20.1756	39.2779
	2	73.4349	9.4463	23.1425	41.0521	7.7777	20.7202	38.2692	5.5570	17.9400	35.2592
	5	54.8848	8.3502	20.5212	36.6424	6.8732	18.3862	34.2068	4.9088	15.9392	31.5773
	10	49.0419	7.9740	19.6142	35.0909	6.5630	17.5771	32.7721	4.6866	15.2434	30.2701
Simply-	0	52.2378	6.6054	21.9520	30.2333	5.3933	20.6110	30.2333	3.8137	19.1764	30.2333
Supported	0.2	42.5584	6.0710	20.2623	29.3532	4.9569	19.0353	29.3532	3.5051	17.7235	29.3532
	0.5	33.6202	5.4981	18.4276	28.3759	4.4892	17.3213	28.3759	3.1743	16.1392	28.3759
	1	25.6662	4.8982	16.4836	27.2821	3.9994	15.5021	27.2821	2.8280	14.4542	27.2821
	2	19.0255	4.3033	14.5346	26.0467	3.5136	13.6758	26.0467	2.4845	12.7594	26.0467
	5	14.1239	3.7864	12.8255	24.6353	3.0916	12.0723	24.6353	2.1861	11.2689	24.6353
	10	12.5953	3.6109	12.2407	23.9232	2.9483	11.5232	23.9232	2.0848	10.7580	23.9232
Clamped-	0	13.3091	2.3316	12.1104	30.2333	1.9382	11.6234	30.2333	1.3987	11.1120	30.1680
Free	0.2	11.7352	2.1682	11.2074	28.8250	1.7796	10.7234	28.4004	1.2304	10.2133	27.9692
	0.5	8.5216	1.9375	10.1517	26.3220	1.6101	9.7476	25.9695	1.1616	9.3236	25.6121
	1	9.0760	1.8240	9.2133	23.7898	1.4332	8.7151	23.3542	0.8087	8.1824	22.9102
	2	4.8000	1.5142	7.9967	20.9685	1.2581	7.6812	20.6955	0.9073	7.3505	20.4189
	5	3.5571	1.3316	7.0529	18.5737	1.1063	6.7757	18.3345	0.7977	6.4850	18.0921
	10	3.1705	1.2697	6.7305	17.7459	1.0548	6.4661	17.5181	0.7606	6.1891	17.2873

Table 4: Effect of the axial force on the first three natural frequencies of (1-1-1) FG sandwich beams for various boundary conditions (Type A).

8. Conclusions

Based on refined shear deformation theory, free vibration of axially loaded FG sandwich beams is presented. Numerical results are obtained for sandwich beams to investigate the critical

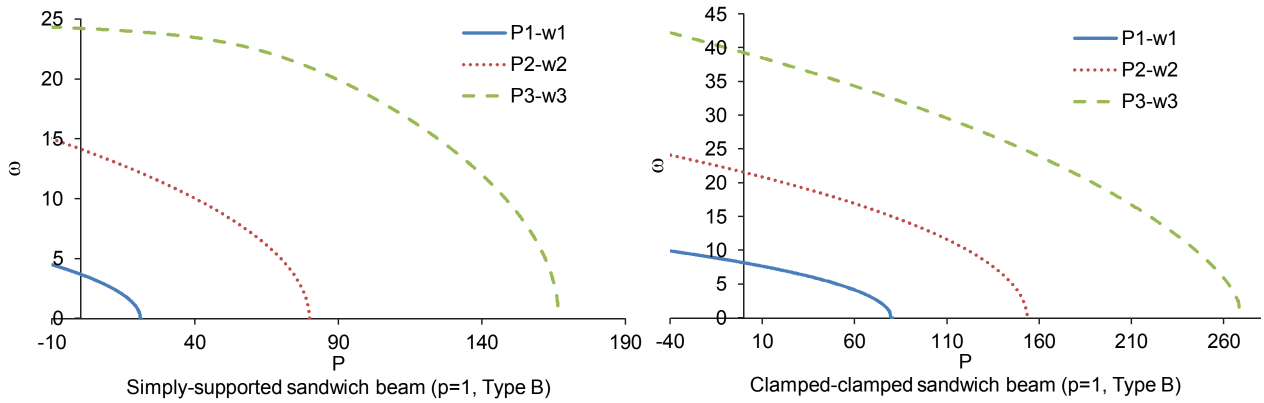

Figure 2: Load-frequency curves of (1-1-1) sandwich beams.

buckling loads, natural frequencies and load-frequency relationship for various power-law index, skin-core-skin thickness ratios and boundary conditions. The present model can provide accurate and reliable results for free vibration of axially loaded FG sandwich beams.

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