A TOP-DOWN ANALYSIS FOR DETERMINING EDGE STRESSES IN COMPOSITE LAMINATES

F. Mujika*, I. Adarraga, M.A. Cantera, J.M. Romera

Materials+Technologies Group, Department of Mechanical Engineering, Polytechnic University College,University of the Basque Country (UPV/EHU) Plaza de Europa 1, 20018 San Sebastián, Spain

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Abstract

A top-down analytic approach is proposed to analyze interlaminar stresses in multidirectional symmetric laminates under tensile load. The coupling effects of each ply are compensated by adjacent plies and the total behavior is similar to an isotropic material, from a deformation point of view. Nevertheless, in-plane and out-of-plane stresses appear in each ply, preventing the coupling deformations. The maximum values of out-of-plane stresses are at the edges of the specimen. In a first step, a half of a symmetric laminate is considered for determining forces and moments per unit length that prevent the global deformation. In a second step, each ply is analyzed with the data obtained from the previous sublaminate.

1. Introduction

According to the laminated plates theory, coupling effects between membrane and plate behaviors do not occur in the case of symmetric laminates. In many cases, coupling effects are related to the generation of shear strains under the application of axial loads. In the membrane case, normal strains are uniform and the generated shear strains have uniform distribution. When normal shear-coupling occurs in the plate behavior, as the distribution of normal strains is linear, induced strains have also linear distribution and then, twisting curvatures appear [1].

The freedom of a deformation mode is related to the absence of stresses relative to that deformation modes. In a opposite manner, when deformation modes are constrained, stresses must act in order to impose the constraint of deformation. Therefore, when in a symmetric laminate membrane-plate coupling does not exist, stresses must adopt the values for constraining deformations.

In the present work, the analysis carried out for a symmetric angle-ply laminate [1] is generalized for symmetric laminates and applied to a quasi-isotropic laminate subjected to a uniform tensile load. In this case bending and twisting curvatures and shear strains are null. Then, normal strains are uniformly distributed in the thickness of the laminate and stresses appear in order to constraint the tendency of each lamina for shear deformation. In a first step a half of the laminate is considered, determining the bending and twisting moments that must act on this part for obtaining null curvatures. Then, interlaminar shear stresses are determined based on equilibrium equations and the assumption of a displacement field.

2. Analytic approach

2.1. Displacement and strain field

The following displacement field has been assumed:

$$\begin{aligned} u &= u_0(x,y) + z\theta_x(x,y) \\ v &= v_0(x,y) + z\theta_y(x) \\ w &= w_0(x,y) = -y\theta_y(x) \end{aligned} \tag{1}$$

where w is the normal deflection of the laminate and thus the laminate has been considered to be inextensible in z-direction. u and v are the in-plane displacements of the laminate which have been assumed to be linear functions of the coordinate z. θ_y is the twisting angle of the laminate. According to Eq. $(1)_3$, it is assumed that specimen remains straight along the width. This assumption is based on the fact that the length-to –width ratio of the specimen is considered to be great. From Eqs (1) the components of the strain field are:

$$\begin{aligned} \varepsilon_x &= u_{,x} = \varepsilon_x^0 + z\kappa_x \\ \varepsilon_y &= v_{,y} = \varepsilon_y^0 + z\kappa_y \\ \varepsilon_z &= 0 \\ \gamma_{xy} &= \gamma_s = u_{,y} + v_{,x} = \gamma_{xy}^0 + z\kappa_{xy} \\ \gamma_{xz} &= u_{,z} + w_{,x} = \theta_x - y\theta_y' \\ \gamma_{yz} &= v_{,z} + w_{,y} = \theta_y - \theta_y = 0 \end{aligned} \tag{2}$$

where,

$$\begin{aligned} &\varepsilon_x^0 = u_{0,x} \qquad \varepsilon_y^0 = v_{0,y} \qquad \gamma_s^0 = u_{0,y} + v_{0,x} \\ &\kappa_x = \theta_{x,x} \qquad \kappa_y = \theta_{y,y} = 0 \qquad \kappa_{xy} = \kappa_s = \theta_{x,y} + \theta_{y,x} = \theta_{x,y} + \theta_y' \end{aligned} \tag{3}$$

According to Eq. $(2)_5$ it results that $\theta_x = \gamma_{zx} + y\theta_y'$. Differentiating with respect to y and replacing in Eq. $(3)_6$, the twisting curvature is:

$$\kappa_{xy} = \kappa_s = \theta_{x,y} + \theta_{y,x} = \gamma_{zx,y} + 2\theta_y' \tag{4}$$

2.2. Stress field and resultant forces and resultant moments

The stress-strain relations for in-plane components in expanded and abbreviated form without taking into account hygrothermal effects are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_s \end{Bmatrix}_k = \begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{xy} & Q_{yy} & Q_{ys} \\ Q_{xs} & Q_{ys} & Q_{ss} \end{bmatrix}_k \left[\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix} \right] \tag{5}$$

where $\{\sigma\}_k$ are in-plane stresses at lamina *k*, $[Q]_k$ are the reduced stiffness coefficients of lamina *k*, $\{\varepsilon^0\}$ are strains of the middle plane and $\{\kappa\}$ are the curvatures of the middle plane. Force and moment resultants concerning in-plane stress components are given by:

$$\begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xs} & B_{xx} & B_{xy} & B_{xs} \\ A_{xy} & A_{yy} & A_{ys} & B_{xy} & B_{yy} & B_{ys} \\ A_{xs} & A_{ys} & A_{ss} & B_{xs} & B_{ys} & B_{ss} \\ B_{xx} & B_{xy} & B_{xs} & D_{xx} & D_{xy} & D_{xs} \\ B_{xy} & B_{yy} & B_{ys} & D_{xy} & D_{yy} & D_{ys} \\ B_{xs} & B_{ys} & B_{ss} & D_{xs} & D_{ys} & D_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix} \tag{6}$$

The laminate stiffness matrices are:

$$\begin{aligned} [A] &= \sum_{k=1}^{n} [Q]_k \left(z_k - z_{k-1} \right) \\ [B] &= \frac{1}{2} \sum_{k=1}^{n} [Q]_k \left(z_k^2 - z_{k-1}^2 \right) \\ [D] &= \frac{1}{3} \sum_{k=1}^{n} [Q]_k \left(z_k^3 - z_{k-1}^3 \right) \end{aligned} \tag{7}$$

The inverse relation of Eq. (6) is:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_s^0 \\ \kappa_x \\ \kappa_y \\ \kappa_s \end{Bmatrix} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xs} & b_{xx} & b_{xy} & b_{xs} \\ a_{yx} & a_{yy} & a_{ys} & b_{yx} & b_{yy} & b_{ys} \\ a_{sx} & a_{sy} & a_{ss} & b_{sx} & b_{sy} & b_{ss} \\ c_{xx} & c_{xy} & c_{xs} & d_{xx} & d_{xy} & d_{xs} \\ c_{yx} & c_{yy} & c_{ys} & d_{yx} & d_{yy} & d_{ys} \\ c_{sx} & c_{sy} & c_{ss} & d_{sx} & d_{sy} & d_{ss} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_s \\ M_x \\ M_y \\ M_s \end{Bmatrix} \tag{8}$$

Concerning shear stresses the notation of Daniel and Ishai [1] has been used, being $yz = q$, $zx = r$ and $xy = s$. In the case of out-of-plane shear strain components, constitutive relation at ply *k* is:

$$\begin{Bmatrix} \gamma_q \\ \gamma_r \end{Bmatrix}_k = \begin{bmatrix} S_{qq} & S_{qr} \\ S_{qr} & S_{rr} \end{bmatrix}_k \begin{Bmatrix} \tau_q \\ \tau_r \end{Bmatrix}_k \tag{9}$$

As it can be seen in Eqs. $(2)_{5,6}$ γ_r and γ_q do not depend on *z* and therefore, they are the same as the mean values along the thickness $\overline{\gamma}_r$ and $\overline{\gamma}_q$. Consequently, the out-of-plane constitutive equations have been expressed in terms of average values through the thickness:

$$\begin{Bmatrix} \bar{\gamma}_q \\ \bar{\gamma}_r \end{Bmatrix} = \begin{bmatrix} \bar{S}_{qq} & \bar{S}_{qr} \\ \bar{S}_{qr} & \bar{S}_{rr} \end{bmatrix} \begin{Bmatrix} \bar{\tau}_q \\ \bar{\tau}_r \end{Bmatrix} = \frac{1}{4h} \begin{bmatrix} \bar{S}_{qq} & \bar{S}_{qr} \\ \bar{S}_{qr} & \bar{S}_{rr} \end{bmatrix} \begin{Bmatrix} V_q \\ V_r \end{Bmatrix} \tag{10}$$

where $\bar{\tau}_r$ and $\bar{\tau}_q$ are mean shear stresses; V_r and V_q are stress resultants induced by τ_r and τ_q, respectively; and $\bar{S}_{ij}$ are equivalent compliance coefficients. Since according to Eq. $(2)_6$ $\gamma_q = 0$, it results that

$$V_q = -\frac{\bar{S}_{qr}}{\bar{S}_{qq}} V_r \quad \text{and} \quad \bar{\gamma}_r = \frac{V_r}{4h}\left(-\frac{\bar{S}_{qr}^2}{\bar{S}_{qq}} + \bar{S}_{rr} \right) = \frac{\bar{S}_{rr}^*}{4h} V_r \tag{11}$$

2.3.Equilibrium equations

Equilibrium equations in terms of resultant forces and moments can be obtained integrating along the thickness the equilibrium equations concerning stresses [3]. Being $2h$ the thicknes of the laminate, equilibrium equations are:

$$\begin{aligned}
&N_{x,x} + N_{s,y} + \tau_r(h) - \tau_r(-h) = 0 \\
&N_{s,x} + N_{y,y} + \tau_q(h) - \tau_q(-h) = 0 \\
&V_{r,x} + V_{q,y} + \sigma_z(h) - \sigma_z(-h) = 0 \\
&M_{x,x} + M_{s,y} + h\left[\tau_r(h) - \tau_r(-h)\right] - V_r = 0 \\
&M_{s,x} + M_{y,y} + h\left[\tau_q(h) - \tau_q(-h)\right] - V_q = 0
\end{aligned} \tag{12}$$

It is worth noting that interlaminar stresses appear in equilibrium equations. Then, if a sublaminate is isolated from the laminate, it is necessary to take them into account. It would be desirable to isolate sublaminates with null stresses in the upper and lower faces.

2.4. Analysis of the half of a symmetric laminate

The half in thickness of a symmetric laminate is isolated. In general, this sublaminate has all kind of coupling effects. It has not stresses applied on the upper face. With respect to the lower face, interlaminar shear stresses are also null, due to the symmetry of the problem. Then, Eqs. (12) can be applied assuming that $\tau_q(\pm h)$and $\tau_r(\pm h)$are null.

Variations in x have not been considered and transverse loads have not been applied. Then, according to Eq. $(12)_1$ N_s is uniform and as $N_s = 0$ at the edges, N_s vanishes along the width. By analogous reasoning in Eqs.$(12)_2$ it results that $N_y = 0$. Then, the laminate is supposed to be under the action of the N_x known force and M_x, M_y and M_s unknown moments. These moments can be determined as a function of κ_s from the following system of equations, coming from Eq. (8) after imposing $\kappa_x = \kappa_y = \theta'_y = 0$ and Eq. (4):

$$\begin{aligned}
&d_{xx} M_x + d_{xy} M_y + d_{xs} M_s = -c_{xx} N_x \\
&d_{xy} M_x + d_{yy} M_y + d_{ys} M_s = -c_{yx} N_x \\
&d_{xs} M_x + d_{ys} M_y + d_{ss} M_s = \gamma_{r,y} - c_{sx} N_x
\end{aligned} \tag{13}$$

With respect to Eq. $(12)_3$ $\sigma_z(-h)$ is unknown. Otherwise, Eqs.$(12)_{4,5}$ reduce to:

$$\begin{aligned} M_{s,y} &= V_r \\ M_{y,y} &= V_q \end{aligned} \tag{14}$$

Differentiating with respect to y the 6th of Eq. (8), assuming that N_x is uniform along the width, replacing Eqs. (14) and considering $M_{x,y}$ it results:

$$\kappa_{s,y}(1-f) = d_{ys}V_q + d_{ss}V_r \quad f = \frac{1}{\Delta}\left(d_{xy}d_{ys} - d_{yy}d_{xs}\right) \quad \Delta = \begin{vmatrix} d_{xx} & d_{xy} & d_{xs} \\ d_{xy} & d_{yy} & d_{ys} \\ d_{xs} & d_{ys} & d_{ss} \end{vmatrix} \tag{15}$$

Differentiating Eq. (4) with respect to y and taking into account Eq. (11):

$$\overline{\gamma}_{r,yy}\left(1-f\right) = \left(d_{ss} - d_{ys}\frac{\overline{S}_{qr}}{\overline{S}_{qq}}\right)\frac{4h}{\overline{S}^*_{rr}}\overline{\gamma}_r = 4h\frac{d^*_{ss}}{\overline{S}^*_{rr}}\overline{\gamma}_r \tag{16}$$

Eq. (16) can be written as:

$$\overline{\gamma}_{r,yy} - k^2\overline{\gamma}_r = 0 \quad \text{where} \quad k^2 = 4h\frac{d^*_{ss}}{\overline{S}^*_{rr}\left(1-f\right)} \tag{17}$$

The general solution of Eq. (17) is:

$$\overline{\gamma}_r\,(y) = C_1 \sinh ky + C_2 \cosh ky \tag{18}$$

Replacing Eq. (18) in the expression of M_s obtained from Eq. (13) and imposing that $M_s = 0$ at the edges $y = \pm b$, C_1 and C_2 are determined. Then, M_x, M_y and M_s can be known.

2.5. Interlaminar stresses

Equilibrium equations of stresses can be expressed as a function of applied moments and the applied force, that are related to V_r. Then, interlaminar stresses are obtained after integration, imposing interlaminar continuity conditions.

2.4. Equivalent shear stiffness

Equivalent shear stiffness coefficients are obtained equating the strain energy of the actual interlaminar stresses with the energy corresponding to averaged values. In that form, the effect of the variation of the interlaminar stresses along the thickness is included.

3. Top-down analysis in a quasi-isotropic laminate

A quasi-isotropic $[0,45,90,-45]_s$ laminate is subjected to a tensile axial load n_x in the longitudinal direction of the laminate as it can be seen in Figure 1.

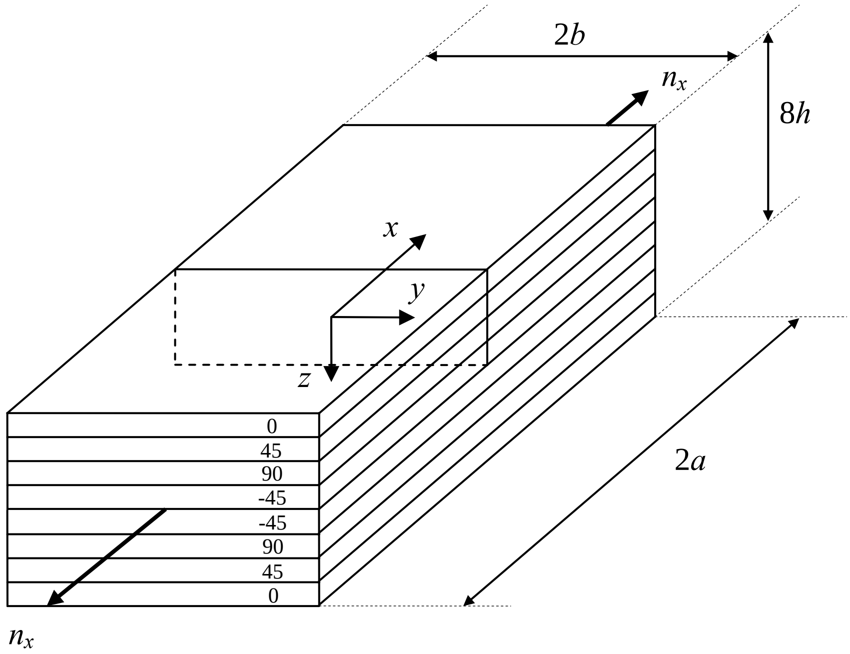

Figure 1. Quasi-isotropic laminate under tensile load.

The analytical procedure for solving this problem has been carried out in two stages. At the first stage, the upper half of the laminate is extracted and analyzed separately. When the upper sublaminate is analyzed, only one half of the axial load is taken into account. This part would present a twisting curvature and two bending curvatures if it were alone. The sublaminate does not actually present any curvature as it is constrained by the twisting moment M_s and the bending moments M_x and M_y induced by the lower sublaminate.

Therefore, the problem has been reduced to a quasi-isotropic sublaminate subjected to an axial force per unit length $N_x^{sub} = n_x/4b$ and unknown twisting and bending moments as it is shown in Figure 2.

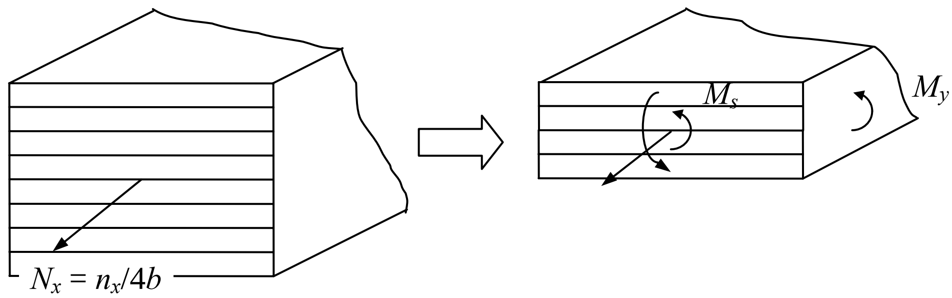

Figure 2. First stage of the top-down approach: extraction of the upper sublaminate.

At the second stage, interlaminar shear stresses are determined according to the previous explanations and the results are compared to those obtained from the Finite Element Method using solid elements and the submodelling technique along the thickness.

4. Conclusions

A top-down analysis is proposed in order to analyze interlaminar shear stresses in symmetric multidirectional laminates. It is applied to a quasi-isotropic laminate in order to determine interlaminar stresses caused by the absence of bending and twisting deformations. Results are compared with those obtained from numerical models of the FEM.

References

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