SOLVING STRESS EQUATIONS IN COMPOSITE CYLINDERS

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ABSTRACT *Cylindrical flywheels under the action of centrifugal forces often are under stress distributions more complex than those of simple disks of short length. Strong edge effects can appear, and these effects vary significantly the distributions. These parts are unusually thick, often unidirectional in the circumferential direction except for a few thin transversal plies. Therefore, the out-of-plane stresses can reach relatively high values, sometimes limiting the speed limit and the energy storage capacity of the flywheel.*
In this paper, we analyze the stresses as a function of axial distance from the edge of the flywheel with the objective of finding an approximated analytical formulation that can predict the distribution of the out-of-plane components. The exact three-dimensional governing equations are integrated, neglecting just a few terms in the differential equations.

1. Introduction

In calculating flywheels composites is common to consider that plane stress conditions are met; the solution found with this hypothesis is accurate only for short length disks or for the center of long ones. To the best of our knowledge, there is no exact and complete solution for flywheels of certain length in which axial tension necessarily has to be taken into account. The problem is solved for cylindrical flywheels of infinite length; see for example Portnov [1], in which the plane strain conditions are assumed, i.e. again uniform stresses along the entire length.
Edge effects in cylinders are included in the solution of Ye [2] for the case of tubes with simple loads, radial pressures in the radial surfaces, but the resulting complex solution is not very useful for flywheels.
The present study is performed in three steps: a) solution for the central uniform area, away from the axial ends, b) variation of the axial stress in the vicinity of the edge, and c) recalculation of the other stresses near the edge.

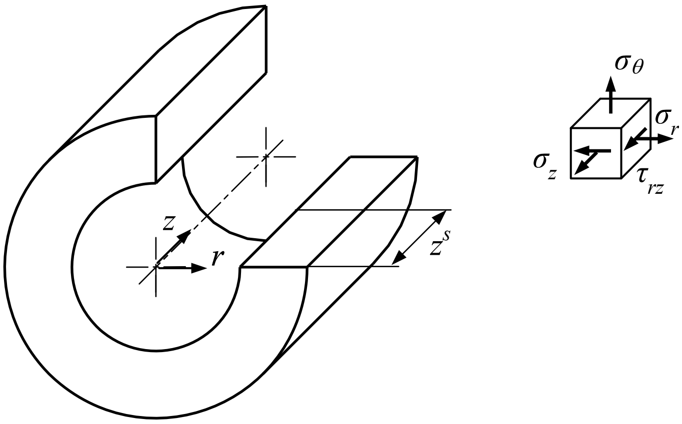

Figure 1. Longitudinal section of the cylindrical rotor and related stresses

2. **Solution far away from edge or uniform zone**

In the zone far away from the axial edge, plane sections should maintain the flat shape after the load is applied and thus axial deformation is constant in all the points, see [1] and [3]. This area will be called "uniform" zone and its axial deformation ε_z^u is constant.
In the equilibrium equation of radial force (with the hypothesis of nullity for shear stress τ_{zr}), the small deformation kinetic equations in circumferential and radial directions along with the material constitutive equation are substituted.

$$\sigma_{r,r} + \frac{\sigma_r}{r} - \frac{\sigma_\theta}{r} + \tau_{zr,z} + \rho\omega^2 r = 0$$

$$\varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_r = u_{,r}$$

$$\begin{Bmatrix} \varepsilon_\theta \\ \varepsilon_r \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{vmatrix} \frac{1}{E_\theta} & -\frac{\nu_{r\theta}}{E_r} & -\frac{\nu_{z\theta}}{E_z} & 0 \\ -\frac{\nu_{\theta r}}{E_\theta} & \frac{1}{E_r} & -\frac{\nu_{zr}}{E_z} & 0 \\ -\frac{\nu_{\theta z}}{E_\theta} & -\frac{\nu_{rz}}{E_r} & \frac{1}{E_z} & 0 \\ 0 & 0 & 0 & \frac{1}{G_{rz}} \end{vmatrix} \begin{Bmatrix} \sigma_\theta \\ \sigma_r \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} \tag{1}$$

The circumferential and radial stresses are function of r. Their expressions include three types of terms, related with: internal force (subindex b), axial deformation ε_z^u (subindex d) and boundary conditions (subindex A and B)

$$\begin{aligned} \sigma_\theta^u &= H_A^u A^u r^{-k_u-1} + H_B^u B^u r^{k_u-1} + H_b^u r^2 + H_{d1}^u + H_{dg}^u \ln r \\ \sigma_r^u &= R_A^u A^u r^{-k_u-1} + R_B^u B^u r^{k_u-1} + R_b^u r^2 + R_{d1}^u + R_{dg}^u \ln r \end{aligned} \tag{2}$$

The transversal effect of these two stresses generates an additional axial stress with smaller but in general non-negligible value. But the resultant force in a section perpendicular to the axis must be zero, allowing the calculation of the axial ε_z.

$$\sigma_z^u = \nu_{z\theta}\sigma_\theta^u + \nu_{zr}\sigma_r^u + E_z\varepsilon_z^u \qquad F_z = \int \sigma_z^u \,\mathrm{d}A = 0 \tag{3}$$

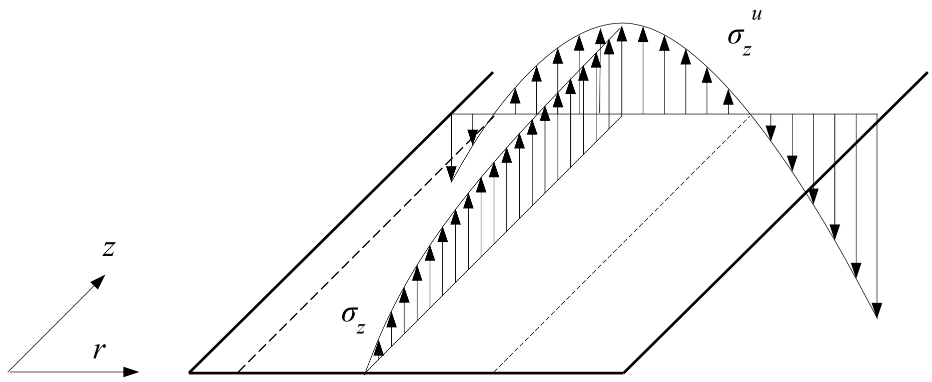

Figure 2. Edge effect on axial stress

3. Solution including edge effect: axial stresses

The axial stress σ_z decreases in the proximity to the axial ends to become zero on the edge itself. On the other hand, the σ_r and σ_θ components created by the centrifugal forces remain fairly constant along z. Therefore, σ_z can be determined with the expression of σ_z^u, equations (1) plus the kinetic equation corresponding to the shear stress τ_{zr}

$$\sigma_{z,z} + \tau_{rz,r} + \frac{\tau_{rz}}{r} = 0 \qquad \gamma_{rz} = .w_{,r} + u_{,z} \tag{4}$$

It is assumed that the contribution of the curvature τ_{zr} /r is is much smaller than the other two.

$$\sigma_{z,zz} + G_{rz}\left(-\frac{\nu_{\theta z}}{E_\theta}\sigma_{\theta,rr} - \frac{\nu_{rz}}{E_r}\sigma_{r,rr} + \frac{1}{E_z}\sigma_{z,rr}\right) + G_{rz}\left(-\frac{\nu_{\theta r}}{E_\theta}\sigma_{\theta,zz} + \frac{1}{E_r}\sigma_{r,zz} - \frac{\nu_{zr}}{E_z}\sigma_{z,zz}\right) = 0 \tag{5}$$

Using the constant assumption of σ_r and σ_θ, the expression can be simplified to another only function of σ_z^u

$$\sigma_{z,zz} + n^2\sigma_z - n^2\sigma_z^u = 0 \qquad \sigma_z = \varphi\,\sigma_z^u \tag{6}$$

The boundary conditions $\sigma_z(z=0)=0$ and $\sigma_{z,z}(z=z^s)=0$ have been considered, where z^s is the half length and the coefficients n and φ are

$$n = \left(\frac{1}{\frac{E_z}{G_{rz}} - \nu_{zr}}\frac{\sigma^u_{z,rr}}{\sigma^u_z}\right)^{0.5} \qquad \varphi = 1 - \frac{e^{jn(z^s-z)} + e^{-jn(z^s-z)}}{e^{jnz^s} + e^{-jnz^s}} \tag{7}$$

4. Solution including edge effect: total stresses

The circumferential and radial stresses for each axial position can be calculated with a process similar to that of the uniform area, but substituting the condition of constant axial strain ε_z^u by the corresponding value of σ_z

$$
\begin{aligned}
\sigma_\theta &= H_A A r^{-k-1} + H_B B r^{k-1} + H_b r^2 + H_{dA} r^{-k_u-1} + H_{dB} r^{k_u-1} + H_{db} r^2 + H_{d1} + H_{dg} \ln r \\
\sigma_r &= R_A A r^{-k-1} + R_B B r^{k-1} + R_b r^2 + R_{dA} r^{-k_u-1} + R_{dB} r^{k_u-1} + R_{db} r^2 + R_{d1} + R_{dg} \ln r
\end{aligned} \quad (8)
$$

where the terms related with boundary conditions and internal forces appear with the same form as in equation (2), but those related with the uniform are different. The latter terms were previously related with the axial deformation ε_z^u now depend on σ_z and therefore σ_z^u, equation (6b), and on both σ_r^u and σ_θ^u from equation (2). The terms are factorized with the functions of the radii to easily study their influence the final results.
The present method of calculation is not only valid for very long rotors, where there is a sufficient distance for the stresses to be uniform, but also the application of the coefficient in equation (7b) is suited to intermediate lengths rotors.

5. Conclusions

- To calculate stresses in flywheels of considerable length, one must resort to approximations that simplify the equations.

- To facilitate the final solution, the problem is solved in two stages: first one solution related to the area far away from the ends and second adding edge effects at the axial ends.

- This separation, although able to simplify the equations, is not sufficient to obtain a relatively simple solution; it is necessary to consider zero some terms.

References

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