MODELLING OF [0/90] LAMINATES SUBJECT TO THERMAL EFFECTS CONSIDERING MECHANICAL CURVATURE AND THROUGH-THE-THICKNESS STRAIN

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Abstract

Non-uniform curvatures and through-the-thickness strain have been incorporated in the Extended Classical Lamination Theory of Dano and Hyer to model curved shapes of bistable laminates. The indication of Tiersten about the use of the sine instead of the tangent function in the derivation of the bending curvature has been introduced. Two expansions of curvatures, named the Mechanical and the Mathematical curvatures, have been developed. Through-the-thickness strain, which is assumed to be uniform in thickness, has been also included in the analysis. Out-of-plane displacements of a grid of points marked at the deformed surface of a carbon/epoxy laminate of square shape have been experimentally obtained using a Coordinate Measuring Machine. Comparison between experimental results of displacements and different analytic approaches has been carried out.

1. Introduction

Cross-ply laminates have an anisotropic response due to elevated temperatures of manufacturing process, and residual thermal stresses lead to a curved shape at Room Temperature. [1]. These laminates can have two stable states at RT: cylindrical state I with a major curvature in x axis and cylindrical state II with an opposite curvature in the orthogonal y axis, as shown in Figure 1.

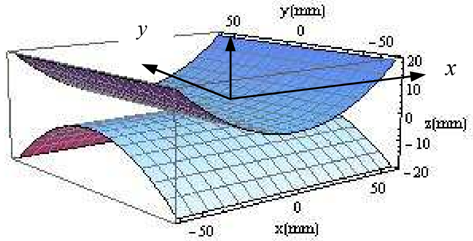

Figure 1. Two stable shapes of a square $[0/90]_T$ laminate

In bistable laminates, one stable shape can change to the other by applying a small amount of energy. This is a geometrically nonlinear phenomenon, known as snap-through. Hyer and co-workers incorporated the geometrical nonlinearities of von Kárman within the Classical Lamination Theory and proposed an Extended Classical Lamination Theory (ECL) [2]. The strain field was approximated by polynomial functions of unknown coefficients using the Rayleigh-Ritz method and by applying the Principle of Minimum Total Potential Energy. This model assumed the hypothesis of uniform curvatures and null through-the-thickness strain. In this work the ECLT is considered as a basic approach. It has been modified by the incorporation of non-uniform curvatures and through-the-thickness strain. Two expressions of bending curvatures are considered [3]: the mathematical expression of curvature that obtained the derivative of the angle from the tangent function, and the mechanical expression of curvature where the derivative of the angle is obtained from the sine function. The terms of second order of slopes are retained in Taylor's polynomial expansions in both cases. Experimental validation of the mentioned approaches is done. Predictions of displacements using different analytic approaches have been compared with experimental results.

2. Basic approach: Extended Classical Lamination Theory (ECLT)

The reference system used is defined in Figure 1, where the origin is located at the geometric centre of the laminate and Ω is the mid-plane of the laminate. In this work square plates of constant thickness h and side-length L are considered. Hygroscopic effects are not taking into account and there is no external forces acting on the laminate. For bi-stablility, the Total Potential Energy (TPE) Π has two minima, each one associated with a stable geometric configuration. In mathematical terms:

$$\frac{\partial \Pi}{\partial e_i} = 0; \quad \text{i=1,n} \quad \text{equilibrium} \quad \frac{\partial^2 \Pi}{\partial e_i \partial e_j} > 0 \quad \text{stability} \tag{1}$$

Where e_i are unknown coefficients estimated by solving a system of nonlinear equations. The function chosen by ECLT to model the out-of-plane displacement is given by:

$$w(x,y,z) = w^0(x,y) = \frac{ax^2 + by^2 + cxy}{2} \tag{2}$$

The mid-plane strains $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0$ and curvatures $\kappa_x, \kappa_y, \kappa_{xy}$, are given by:

$$\begin{aligned} &\varepsilon_x^0 = \frac{\partial u^0}{\partial x} + \frac{1}{2}\left(\frac{\partial w^0}{\partial x}\right)^2, \quad \varepsilon_y^0 = \frac{\partial v^0}{\partial y} + \frac{1}{2}\left(\frac{\partial w^0}{\partial y}\right)^2, \quad \gamma_{xy}^0 = \gamma_s^0 = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \frac{\partial w^0}{\partial x}\frac{\partial w^0}{\partial y}, \\ &\kappa_x = -\frac{\partial^2 w^0}{\partial x^2} = -a, \qquad \kappa_y = -\frac{\partial^2 w^0}{\partial y^2} = -b, \qquad \kappa_{xy} = -2\frac{\partial^2 w^0}{\partial x \partial y} = -c \end{aligned} \tag{3}$$

The strain field $\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa^0\}$ of the basic approach using unknown coefficients is given by:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} d_1 + d_2 x^2 + d_3 xy + d_4 y^2 \\ d_5 + d_4 x^2 - d_3 xy + d_2 y^2 \\ 2d_6 + \left(4d_4 + ab - \frac{c^2}{4}\right)xy + \left(d_7 + \frac{d_3}{2} + \frac{ac}{4}\right)x^2 + \left(d_8 - \frac{d_3}{2} + \frac{bc}{4}\right)y^2 \end{Bmatrix} - z \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} \tag{4}$$

3. Mathematical and mechanical curvatures

3.1. Motivation

Timoshenko, obtained a simplified mathematical expression of curvatures κ_x, and κ_y for small defections assuming that the mid-plane was undeformed. The angle can be replaced by the tangent function. But in composite laminates, the mid-plane Ω is deformed. The longitudinal strain ε_x at z distance measured from the mid-plane of a deformed element in the zx plane is:

$$\varepsilon_x = \frac{(\rho_x + z)d\theta_x - ds_x}{ds_x} = \frac{ds'_x - ds_x}{ds_x} + z\frac{d\theta_x}{ds_x} = \varepsilon_x^0 + \frac{z}{\rho_x} \tag{5}$$

Where: ds'_x is the length of a deformed line of the mid-plane in the zx plane; ρ_x is the radius of curvature at the mid-plane Ω, being $ds'_x = \rho_x d\theta_x$; and ds_x is the undeformed length, being $ds_x = dx$.

The mathematical expression of the bending curvature is based on a function w that depends on the independent variable $w = w(x)$. Each x has a w value in its vertical line. In this case, dw and dx are related by the tangent function. The curvature obtained by this way will be named mathematical curvature.

In bending of composites beams and plates, the vertical displacement w is related also to the undeformed coordinate x. Nevertheless, the displacement is not in the vertical line of the corresponding x coordinate due to horizontal displacements u^0, according to Tiersten [4].

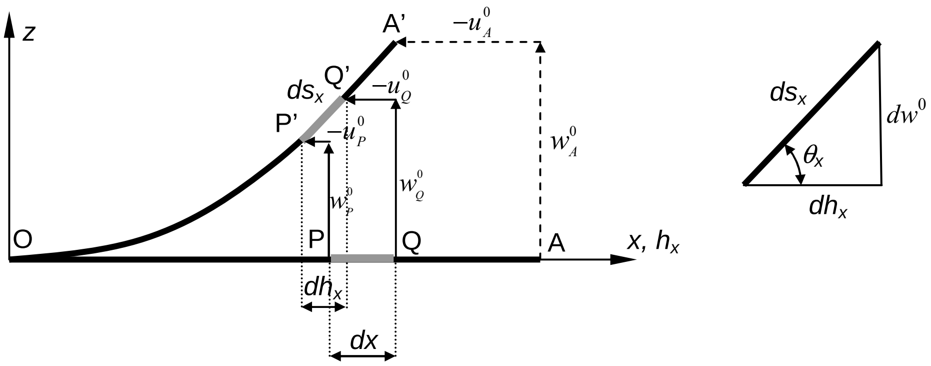

Figure 2. Deformed and undeformed configurations of a level curve of the laminate in the *zx* plane.

Figure 2 shows half of a level curve in the reference plane of a plate with great slopes. Unprimed letters represent positions in the undeformed state and primed letters represent positions in the deformed state. The initial element PQ of length dx in the undeformed state moves to the element P'Q' in the deformed state. It is assumed that $ds'_x = ds_x = dx$. Otherwise, the length of *dx* is not the same that the horizontal projection of the P'Q' segment, dh_x. In this case, *dw* and *dx* are related by the sin function [3]. The curvature obtained by this way will be named mechanical curvature.

3.2. Mathematical curvature

According to Figure 2, the angle θ_x is $\theta_x = \arctan\left(\frac{\partial w}{\partial h_x}\right)$. Differentiating θ_x with respect to s_x:

$$\frac{d\theta_x}{ds_x} = \frac{d\theta_x}{dh_x}\frac{dh_x}{ds_x} = \frac{\frac{\partial^2 w}{\partial h_x^2}}{\left(1+\left(\frac{\partial w}{\partial h_x}\right)^2\right)^{(3/2)}} \tag{6}$$

Since in the mathematical curvature is assumed that $dh_x = dx$ the curvature κ_x is given by:

$$\kappa_x = \frac{1}{\rho_x} = -\frac{d\theta_x}{ds_x} = -\frac{\frac{\partial^2 w}{\partial x^2}}{\left[1+\left(\frac{\partial w}{\partial x}\right)^2\right]^{(3/2)}} \tag{7}$$

Making a Taylor's series expansions in Eq. (7) and retaining the first term, considering the out of plane displacement w of the Eq. the curvatures κ_x and κ_y can be expressed as:

$$\begin{aligned}
\kappa_x &= -\frac{\partial^2 w}{\partial x^2}\left[1-\frac{3}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = -a+\frac{3a^3}{2}x^2+\frac{3ac^2}{8}y^2+\frac{3a^2c}{2}xy \\
\kappa_y &= -\frac{\partial^2 w}{\partial y^2}\left[1-\frac{3}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] = -b+\frac{3b^3}{2}y^2+\frac{3bc^2}{8}x^2+\frac{3b^2c}{2}xy
\end{aligned} \tag{8}$$

The strain field is given by:

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix} d_1+d_2x^2+d_3xy+d_4y^2 \\ d_5+d_4x^2-d_3xy+d_2y^2 \\ 2d_6+\left(4d_4+ab-\frac{c^2}{4}\right)xy+\left(d_7+\frac{d_3}{2}+\frac{ac}{4}\right)x^2+\left(d_8-\frac{d_3}{2}+\frac{bc}{4}\right)y^2 \end{Bmatrix} - z\begin{Bmatrix} +a-\frac{3a^3}{2}x^2-\frac{3ac^2}{8}y^2-\frac{3a^2c}{2}xy \\ +b-\frac{3b^3}{2}y^2-\frac{3bc^2}{8}x^2-\frac{3b^2c}{2}xy \\ c \end{Bmatrix} \tag{9}$$

The above unknown parameters (*a,b,c* and from d_1 to d_8) were determined by Eq. (1).

3.3. Mechanical curvature

In the case of mechanical curvature of composites, the value of θ_x is obtained from the sine function because ds_x is known, $\theta_x = \arcsin\left(\frac{\partial w}{\partial s_x}\right)$ Differentiating θ_x with respect to s_x it results:

$$\frac{d\theta_x}{ds_x} = \frac{\frac{\partial^2 w}{\partial s_x^2}}{\sqrt{1-\left(\frac{\partial w}{\partial s_x}\right)^2}} \tag{10}$$

Introducing the condition $ds_x = dx$ and including the change of sign, the curvature is given by:

$$\kappa_x = \frac{1}{\rho_x} = -\frac{d\theta_x}{ds_x} = -\frac{d\theta_x}{dx} = -\frac{\frac{\partial^2 w}{\partial x^2}}{\sqrt{1-\left(\frac{\partial w}{\partial x}\right)^2}} \tag{11}$$

Making a Taylor series expansion of Eq. (11) and retaining the first term, κ_x and κ_y are:

$$\kappa_x = -\frac{\frac{\partial^2 w}{\partial x^2}}{\sqrt{1-\left(\frac{\partial w}{\partial x}\right)^2}} = -\frac{\partial^2 w}{\partial x^2}\left[1+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \qquad \kappa_y = -\frac{\frac{\partial^2 w}{\partial y^2}}{\sqrt{1-\left(\frac{\partial w}{\partial y}\right)^2}} = -\frac{\partial^2 w}{\partial y^2}\left[1+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] \tag{12}$$

Assuming the out of plane displacement w of the Eq. the curvatures κ_x and κ_y are given by:

$$\begin{aligned} \kappa_x &= -\frac{\partial^2 w}{\partial x^2}\left[1+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = -a - \frac{a^3}{2}x^2 - \frac{ac^2}{8}y^2 - \frac{a^2c}{2}xy \\ \kappa_y &= -\frac{\partial^2 w}{\partial y^2}\left[1+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] = -b - \frac{b^3}{2}y^2 - \frac{bc^2}{8}x^2 - \frac{b^2c}{2}xy \end{aligned} \tag{13}$$

The strain field is given by [3]:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} d_1 + d_2x^2 + d_3xy + d_4y^2 \\ d_5 + d_4x^2 - d_3xy + d_2y^2 \\ 2d_6 + \left(4d_4 + ab - \frac{c^2}{4}\right)xy + \left(d_7 + \frac{d_3}{2} + \frac{ac}{4}\right)x^2 + \left(d_8 - \frac{d_3}{2} + \frac{bc}{4}\right)y^2 \end{Bmatrix} - z\begin{Bmatrix} a + \frac{a^3}{2}x^2 + \frac{ac^2}{8}y^2 + \frac{a^2c}{2}xy \\ b + \frac{b^3}{2}y^2 + \frac{bc^2}{8}x^2 + \frac{b^2c}{2}xy \\ c \end{Bmatrix} \tag{14}$$

In $[0/90]_T$ laminates the twisting curvature is null, thus the parameter c=0 Figure 3 shows κ_x mathematical and mechanical curvatures plotted versus L/h at the centre (x=y=0) the total thickness h varying from 0.25 to 1 mm. They are compared with the uniform curvatures of the basic approach. Square $[0/90]_T$ laminates with layers of the same thickness are considered. The material is carbon/epoxy AS4/8552, its properties have been obtained from [5].

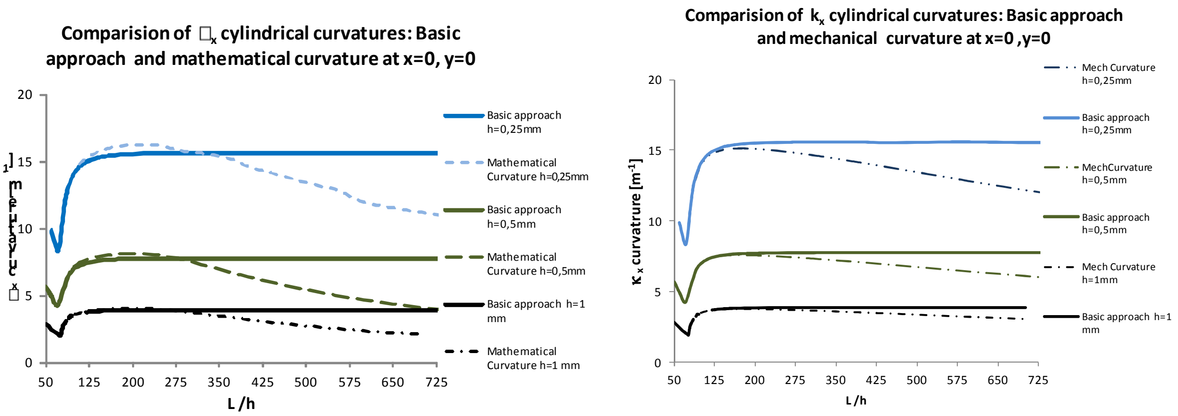

Figure 3. Deformed and undeformed configurations of a level curve of the laminate in the *zx* plane.

4. Full approach: mechanical curvature and uniform through-the-thickness strain.

Total strains of the full approach proposed, considering a uniform through-the-thickness strain ε_z=are given by [3]:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} d_1 + d_2 x^2 + d_3 xy + d_4 y^2 \\ d_5 + d_4 x^2 - d_3 xy + d_2 y^2 \\ d \\ 2d_6 + \left(4d_4 + ab - \dfrac{c^2}{4}\right)xy + \left(d_7 + \dfrac{d_3}{2} + \dfrac{ac}{4}\right)x^2 + \left(d_8 - \dfrac{d_3}{2} + \dfrac{bc}{4}\right)y^2 \end{Bmatrix} - z \begin{Bmatrix} a + \dfrac{a^3}{2}x^2 + \dfrac{ac^2}{8}y^2 + \dfrac{a^2 c}{2}xy \\ b + \dfrac{b^3}{2}y^2 + \dfrac{bc^2}{8}x^2 + \dfrac{b^2 c}{2}xy \\ 0 \\ c \end{Bmatrix} \quad (15)$$

There are 12 unknown coefficients to be determined by Eq.(1). A comparison of the κ_x curvature predicted by the basic approach taking into account ε_z and the full approach is shown in Figure 4, for thickness 0.25 and 0.5 mm. Two parts are identified:

- The surrounds of bifurcation point. The effect of considering ε_z is to increase the length associated with the bifurcation point.
- Stable branch of the curvature, where the curvature at the centre obtained by the full approach is lesser than the one obtained by considering only the mechanical curvature. Moreover, both of them decrease when the ratio L/h increases.

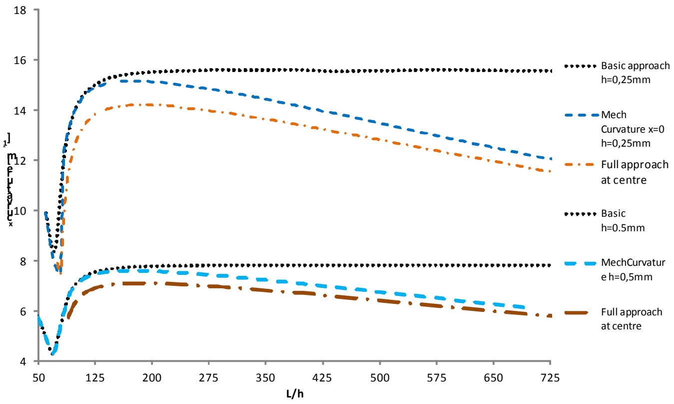

Figure 4. Comparison of curvatures predicted by different models basic, mechanical and Full approach with h=0.25 mm; 0.5 mm

5. Experimental validation.

$[0_2/90_2]_T$ laminates of T6T/F593 carbon/epoxy unidirectional prepreg were manufactured in a hot press, according [6] being L/h = 337. A grid 10 x 10 mm^2 was drawn on the deformed surface. Measuring points are equidistant Δs = 10 mm in the curved drawn lines and columns. Each curved segment Δs has an associatedΔh_x, that is maximum at the centre of the line (x=0) and minimum near the border due to the curved shape. The composite was keep in an oven at 120ºC during 72 hours order to avoid hygroscopic effects, and it was measured by a Coordinate Measuring Machine (CMM) according to [6]. The x coordinate provided the projected coordinate h_x and the z coordinate provided the out-of-plane displacement w.

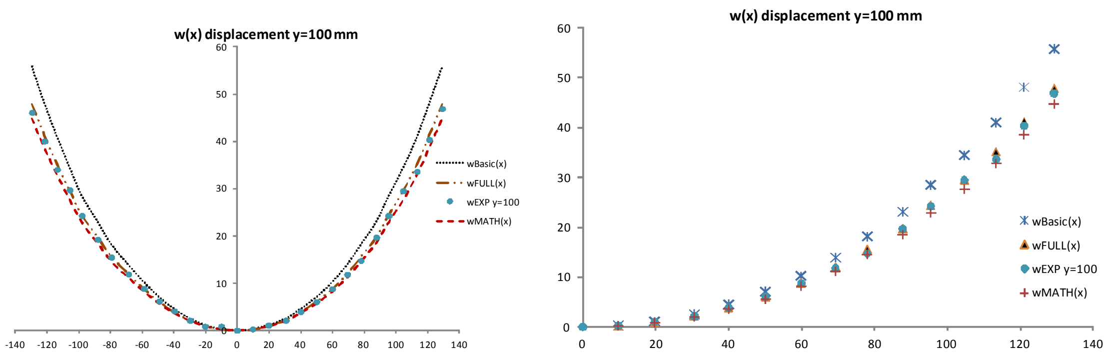

Figure 5: Comparison of experimental out-of- plane displacements and those obtained from basic, mathematical and full approaches in the line y=100 mm.

In Figure 5 out-of-plane displacements obtained experimentally are compared with those obtained from basic, mathematical and full approaches. It can be stated that the differences between the full approach and the mathematical approach are due to the different analytic expressions of curvatures.

The results of the basic approach are greater than the experimental ones. The mathematical and full approaches fit experimental data, being lesser the values obtained from the mathematical approach. It is expected that differences would increase for greater values of L/h.

6. Conclusions.

The present approach introduces non uniform curvatures and through-the-thickness strain in the ECLT of Hyer to model curved shapes of bistable laminates of initial square shape.

Exact bending curvatures are obtained in the mathematical sense and in the mechanical sense. The mechanical curvature seems to be more suitable for studying bending problems of beams and plates, as it is derived based on the fact that horizontal displacements must be considered in addition to vertical displacements.

The influence of considering a uniform value of through-the-thickness strain ε_z was noticeable in the surrounds of bifurcation point: the length-to-thickness ratio associated with bifurcation point increases. Furthermore, the value of cylindrical curvatures decrease.

Analytic approaches have been compared with experimental results of displacements obtained for a square carbon/epoxy laminate with L/h = 337. The approaches that take into account non-uniform curvatures fit better experimental data that the basic approach. The displacements obtained from the approach based on mechanical curvatures are greater than those obtained from mathematical curvatures. The fitting with experimental data do not allow to state that one of them is much better than the other from a experiment-approach comparison point of view. Nevertheless, as stated before, the mechanical curvature is considered more suitable based on arguments related to the kinematics of deformation.

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