FINITE FRACTURE MECHANICS VS. COHESIVE CRACK MODELLING: AN ANALYTICAL COMPARISON BASED ON CASE STUDIES

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Abstract

In recent years, Finite Fracture Mechanics has proven to be an effective tool to predict the strength of mechanical components. Since the proposed method rests on a linear elastic solution, it allows fast strength predictions suitable for preliminary sizing and optimization of plain or composite structures. While several contributions in the Scientific Literature have proven the soundness of Finite Fracture Mechanics by means of a comparison with experimental data, in the present paper we intend to corroborate the approach by showing that usually failure load predictions are very close to the ones provided by the widely-spread and well-established Cohesive Crack Model.

1. Introduction

The Cohesive Crack Model (CCM) allows one to obtain accurate and physically-based strength predictions in plain or composite structural elements with stress concentrations or stress intensifications. Unfortunately, CCM usually requires a numerical implementation with large computing times that are not acceptable for preliminary sizing of structural details.

Fast strength predictions can be obtained by applying the point stress (PS) criterion (or the average stress criterion). These methods predict failure when the stress at (or over) a certain distance (the so-called critical distance) reaches the material tensile strength. Nevertheless these approaches show some drawbacks [1], mainly due to the fact that the critical distance is not a material property, thus requiring expensive experimental programs to identify the critical distances for different materials and geometries [2]. On the other hand, the recently introduced Finite Fracture Mechanics (FFM) allows one to overcome this shortcoming since the length of the critical distance is an outcome of the structural problem [3,1,4]. Furthermore FFM possesses a clear physical interpretation, i.e. fracture is supposed to propagate by finite steps. Thus, in the authors' opinion, FFM can be seen as the right candidate criterion to achieve accurate, physically-based and fast strength predictions.

Aim of the present paper is to corroborate this choice by showing that, for a couple of case studies, the CCM and the FFM strength predictions are in a very good agreement. The two example problems herein considered are represented by an infinite slab with: (*i*) a re-entrant corner; (*ii*) a short crack. In both cases, we see a transition from a toughness-governed failure to a stress-governed one, as the notch opening angle ω increases in the former case, and as the crack length decreases in the latter case. The examples are both solved analytically.

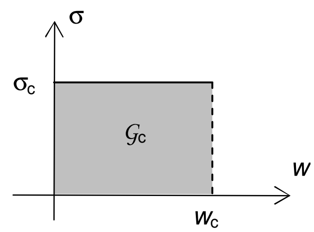

Figure 1. Dugdale cohesive law.

In order to achieve an analytical solution for both the geometries, we will assume a Dugdale shape for the cohesive law, i.e. a stress constant and equal to σ_c as the crack opening w is comprised between 0 and w_c (hence the fracture energy $\mathcal{G}_c$ is given by $\sigma_c \times w_c$, see fig. 1), and a point-wise stress requirement for the FFM criterion [3]. Thus, according to FFM, failure occurs whenever the normal stress over the crack increment Δ (the crack is supposed to propagate along the x axis) is larger than σ_c and, contemporaneously, the energy available for the finite crack increment is larger than $\mathcal{G}_c \times \Delta$ (a being the crack length):

$$\begin{cases} \sigma_y(x) \geq \sigma_c, \qquad 0 < x < \Delta \\ \int_a^{a+\Delta} \mathcal{G}(a')\, \mathrm{d}a' \geq \mathcal{G}_c \Delta \end{cases} \tag{1}$$

2. Re-entrant corner

As a first case, let us consider a V-notched structure (fig. 2a) under mode I loading conditions, ω being the notch opening angle. Since Williams' work, it is known that the stress field is singular. The order of singularity is (1–λ), where the eigenvalue λ is comprised between 0.5, for $\omega = 0$, and 1, for $\omega = \pi$, when the singularity disappears (straight edge). Hence, except for the extreme cases, the stress field is singular, but with a power of the singularity less than 1/2. Therefore, both simple stress criteria and LEFM fail in predicting the strength of V-notched components, providing respectively null or infinite failure loads.

The asymptotic stress field is univocally characterized by the Generalized Stress Intensity Factor (GSIF) K_I^*. In his pioneering paper, Carpinteri [5] proposed to correlate the failure load with the critical value of K_I^*, i.e. the generalized fracture toughness K_{Ic}^*. In the following we show that both CCM and FFM corroborate this conjecture, furthermore providing an expression relating the generalized fracture toughness to the tensile strength and the fracture toughness.

2.1. Cohesive Crack Model

The V-notch problem has been faced by means of the CCM by Henninger et al. [6] through an asymptotic matching approach and by Shi [7] using suitable path-independent integrals. However we will show that the solution can be achieved more simply by exploiting some shape functions available in the Literature as well as some analytical results.

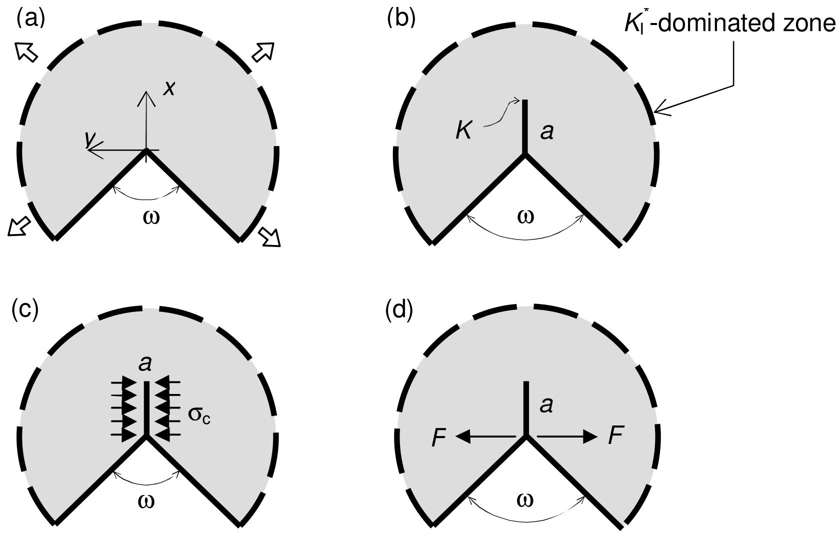

Figure 2. Re-entrant corner under mode I loading: reference system (a); V-notch emanated crack (b); V-notch emanated-crack loaded by a constant stress field (c); V-notch emanated crack loaded by a pair of opening forces at the V-notch tip (d).

For a crack stemming from a V-notch tip (fig. 2b) falling within the K_I^*-dominated zone, simple dimensional analysis arguments yield to the following dependence of the SIF upon the crack length a, the notch opening angle ω and the GSIF:

$$K_I = \mu(\omega) K_I^* a^{\lambda - 1/2} \tag{2}$$

Highly accurate discrete μ values can be found in [8,9]. Analytical interpolating expressions μ(ω) can be found in [10,11] by manipulating preliminary results provided in [12,13].

According to Dugdale model, a process zone appears ahead the V-notch vertex. In the process zone the stress is constant and equal to the critical stress σ_c (see fig. 2c). Such a stress distribution generates a SIF equal to:

$$K_I = -\gamma(\omega)\, \sigma_c \sqrt{\pi a} \tag{3}$$

The shape function γ(ω) is provided in [12] with an accuracy better than 1%. The length a_p of the process zone is thus determined by the condition of a vanishing SIF at the crack tip, i.e.:

$$\mu(\omega) K_I^* a_p^{\lambda - 1/2} - \gamma(\omega)\, \sigma_c \sqrt{\pi a_p} = 0 \tag{4}$$

leading to:

$$a_p = \left(\frac{\mu K_I^*}{\gamma\, \sigma_c \sqrt{\pi}} \right)^{\frac{1}{1-\lambda}} \tag{5}$$

It is worth observing that eqn (5) contains, as special cases $a_p = (\pi/8)(K_I/\sigma_c)^2$ for a crack and $a_p = (\sigma_\infty/\sigma_c)^\infty$ for a flat edge, $K_I^* = \sigma_\infty$ being the remote tensile stress. This last expression is coherent, providing a null process zone for $\sigma_\infty < \sigma_c$ and an infinite one for $\sigma_\infty > \sigma_c$.

Increasing the external load, K_I^* will grow proportionally, and, consequently, also the process zone will increase (with a power law of the load equal or larger than 2). According to the CCM terminology, at the distance a_p from the V-notch vertex the *fictitious* crack tip is placed, since the stress among crack lips is not zero but equal to σ_c. The corresponding crack mouth opening displacement (CMOD) can be easily computed starting from the SIF for a pair of forces F (see fig. 2d), whose value is known analytically (exact solution; see [12]):

$$K_I = \beta(\omega)\,\frac{F}{\sqrt{a}} \tag{6}$$

A straightforward application of Castigliano's theorem allows one to compute the CMOD w as (where $E' = E/(1-\nu^2)$, E being the Young modulus and ν the Poisson ratio):

$$w = \frac{2}{E'}\int_0^{a_p}\left[K_I(K_I^*, a') + K_I(\sigma_c, a')\right]\frac{\partial K_I(F,a')}{\partial F}\,\mathrm{d}a' = \frac{2\beta}{E'}\frac{1-\lambda}{\lambda}\left[\frac{\mu K_I^*}{(\gamma\sqrt{\pi}\,\sigma_c)^{\lambda}}\right]^{\frac{1}{1-\lambda}} \tag{7}$$

The *real* crack tip will appear at the V-notch vertex only when the CMOD reaches its critical value w_c (see fig. 1). At that point the interaction between the crack lips vanishes and the structure reaches the maximum sustainable load. Since $w_c = K_{Ic}^2/(E'\sigma_c)$, eqn (7) provides the generalized fracture toughness according to the CCM as:

$$K_{Ic}^* = \xi_{CCM}(\omega)\,\sigma_c\, l_{ch}^{1-\lambda} \tag{8}$$

where l_{ch} is Irwin's length $(K_{Ic}/\sigma_c)^2$ and the dimensionless coefficient ξ_{CCM} is given by:

$$\xi_{CCM}(\omega) = \frac{(\gamma\sqrt{\pi})^{\lambda}}{\mu}\left[\frac{\lambda}{2\beta(1-\lambda)}\right]^{1-\lambda} \tag{9}$$

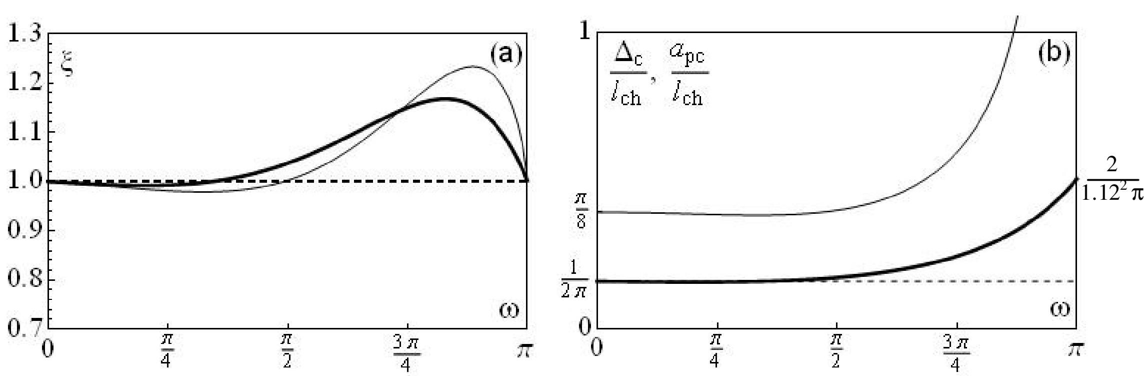

Figure 3. Dimensionless coefficient ξ vs. notch opening angle ω (a) according to: FFM (thick line), CCM (thin line), PM (dashed line). Normalized crack advancement for FFM (thick line), for PM (dashed line) and process zone size for CCM (thin line) vs. notch opening angle ω (b).

Thus, by means of eqn (8), CCM proves that the GSIF can be effectively used to correlate failure of V-notched components. Furthermore, eqn (8) shows the dependence of the generalized fracture toughness on the tensile strength and the fracture toughness. It is easy to check that ξ_{CCM} is equal to unity for $\omega = 0,\pi$, so that, as expected, the generalized fracture toughness equals the fracture toughness and the tensile strength for a crack and a flat edge, respectively. The dependence of the parameter ξ_{CCM} on the notch opening angle ω is drawn in fig.3a. On the other hand, substitution of eqn (8) into eqn (5) provides the expression of the process zone length a_{pc} at critical conditions:

$$a_{pc} = \frac{\lambda}{2\beta\gamma\sqrt{\pi}\,(1-\lambda)}\, l_{ch} \tag{10}$$

which varies from the well-known Dugdale plastic zone estimate $(\pi/8)\ l_{ch}$ for a crack to infinite for a flat edge (see fig. 3b).

2.2. Finite Fracture Mechanics

In order to apply the FFM criterion (1), we need the stress field ahead the V-notch tip along the notch bisector. After Williams, it reads:

$$\sigma_y = \frac{K_I^*}{(2\pi r)^{1-\lambda}} \tag{11}$$

It is easy to prove [3] that, for positive geometries, the lowest load satisfying the two inequalities in eqn (1) is achieved when they are strictly fulfilled, i.e. when:

$$\begin{cases} \sigma_y(\Delta) = \sigma_c \\ \int_0^\Delta K_I^2 \mathrm{d}a' = K_{Ic}^2 \Delta \end{cases} \tag{12}$$

where $\mathcal{G}_c = K_{Ic}{}^2/E'$ has been used. Upon substitution of eqns (11) and (2), eqn (12) becomes:

$$\begin{cases} \dfrac{K_I^*}{(2\pi\Delta)^{1-\lambda}} = \sigma_c \\ (\mu\ K_I^*\ \Delta^\lambda)^2 = 2\lambda\ K_{Ic}^2 \Delta \end{cases} \tag{13}$$

Such a system provides the value of the finite crack advance Δ_c as well as the generalized fracture toughness $K_{Ic}{}^*$:

$$\Delta_c = \frac{2\lambda}{\mu^2 (2\pi)^{2(1-\lambda)}} l_{ch} \tag{14a}$$

$$K_{Ic}^* = \xi_{FFM}(\omega)\ \sigma_c\ l_{ch}^{1-\lambda} \tag{14b}$$

The dimensionless coefficient ξ_{FFM} is now given by:

$$\xi_{FFM}(\omega) = \left[\frac{2\lambda}{\mu^2 (2\pi)^{1-2\lambda}}\right]^{1-\lambda} \tag{15}$$

which is plotted in fig. 3a. It is thus evident the fairly good agreement between the FFM approach (in its point-wise stress version) and CCM (when described by a Dugdale cohesive law), whereas PM predictions are pretty far from the other models. Finally it is worth observing that, although the absolute values of the finite crack extension and the process zone do not match each other, their trend with respect to the notch opening angle (see fig.3b) is the same.

3. Short cracks

The second case we are considering is an infinite slab with a central crack of length $2a$ under a remote uni-axial stress σ orthogonal to the crack (see fig.4). A completely analytical solution for both the models is achievable for this simple geometry.

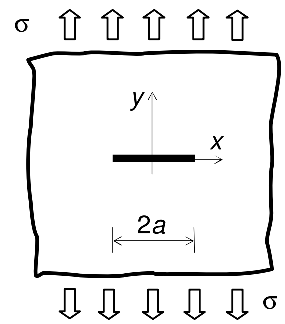

Figure 4. A central through crack in an infinite slab loaded orthogonally to the crack.

3.1. Cohesive Crack Model

The original Dugdale work aimed to get an estimate of the plastic zone ahead of a crack in sheets under a sufficiently small remote tensile stress, which is equal to the well-known value $(\pi/8)\, l_{ch}$ at incipient failure. However an exact value of the plastic zone size can be obtained analytically also when the remote stress approaches the yield stress σ_c, i.e. for short cracks (see e.g. [14]). In such a case, at incipient failure, the plastic (or process) zone size is equal to:

$$a_{pc} = a\left(e^{\frac{\pi\, l_{ch}}{8\, a}} - 1\right) \tag{16}$$

The corresponding failure stress σ_f is:

$$\sigma_f = \sigma_c \frac{2}{\pi} \arccos\left(e^{-\frac{\pi\, l_{ch}}{8\, a}}\right) \tag{17}$$

Note that, for sufficiently large cracks (i.e. $l_{ch}/a \rightarrow 0$), eqn (16) and (17) provide the Dugdale estimate and the LEFM failure stress $K_{Ic}/\sqrt{(\pi a)}$, respectively. On the other hand the process zone tends to infinite and the failure stress to σ_c if $a \rightarrow 0$ (see fig. 5).

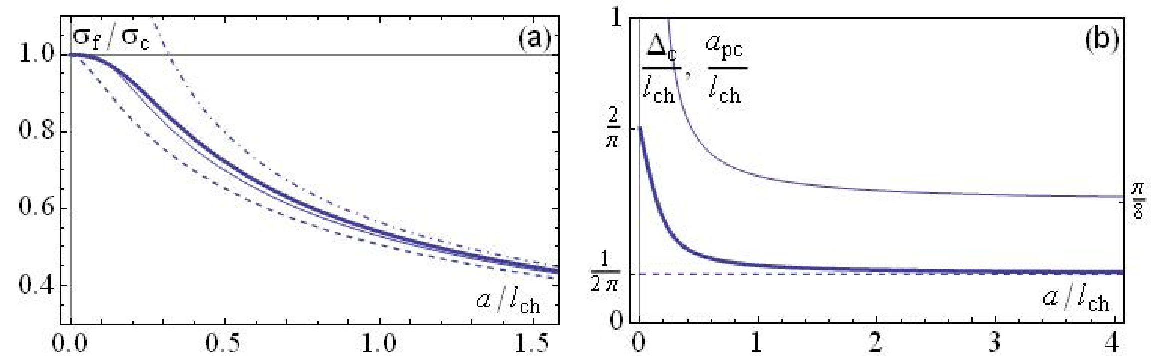

Figure 5. Normalized failure stress vs. normalized crack length (a): FFM (thick line), CCM (thin line), PM (dashed line), LEFM (dot-dashed line). Normalized crack advancement for FFM (thick line), for PM (dashed line) and process zone size for CCM (thin line) vs. normalized crack length (b).

3.2. Finite Fracture Mechanics

To apply the FFM criterion, we need the stress field ahead the crack tip as well as the SIF. They read, respectively:

$$\sigma_y(x) = \frac{x}{\sqrt{x^2 - a^2}}\sigma \tag{18a}$$

$$K_I(a) = \sigma\sqrt{\pi a} \tag{18b}$$

Upon substitution of eqns (18) into the FFM system (12) and by integration, we get:

$$\begin{cases} \left(\dfrac{\sigma}{\sigma_c}\right)^2 = 1 - \left(\dfrac{a}{a+\Delta}\right)^2 \\ \left(\dfrac{\sigma}{\sigma_c}\right)^2 = \dfrac{2/\pi \;\, l_{ch}}{2a+\Delta} \end{cases} \tag{19}$$

The solution of the system provides the finite crack extension Δ_c as the solution of the following third order algebraic equation:

$$\Delta_c (2a + \Delta_c)^2 = \frac{2}{\pi} l_{ch} (a + \Delta_c)^2 \tag{20}$$

Eqn (20) has only one positive, real solution, which can be expressed analytically. Once Δ_c is obtained, the failure stress σ_f is obtained by substitution of Δ_c into either the first or the second equation of the system (19). Taking the first one, we get:

$$\frac{\sigma_f}{\sigma_c} = \frac{\sqrt{\Delta_c(2a + \Delta_c)}}{a + \Delta_c} \tag{21}$$

Equations (20) and (21) are plotted in fig. 5. It is evident the almost perfect agreement between the failure stress estimates provided by the FFM approach and CCM. Both the models approach the σ_c value for $a \to 0$ with a flat tangent and the LEFM estimate for $a \to \infty$. Predictions based on the simpler PM are relatively far from the FFM and CCM ones. Also in this latter case, although the absolute values of the finite crack extension and the process zone are pretty far from each other, the trend with respect to the crack size is almost identical.

4. Conclusion

For a couple of case studies, we have shown that, assuming a perfectly plastic rectangular cohesive law (Dugdale-type) and a point-wise stress condition for the FFM criterion, the strength predictions provided by CCM and FFM are in close agreement. Since FFM is much easier to apply, these results corroborate the use of FFM as an effective tool for preliminary sizing and optimization of structural components.

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