WEIGHT OPTIMISATION OF AN ELLIPTICAL CROSS-SECTION OF A SANDWICH FUSELAGE

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Abstract
Most pressurised aircraft possess a circular fuselage because it can balance pressure by tensile loading. Non-circular fuselages offer flexibility in optimising space for passenger compartment but pressure loads induce bending moments adding stresses in fuselage panels. Classical thin-skin constructions require reinforcements to bear pressurisation loads in these shapes. On the other hand, sandwich constructions are more effective at bearing bending moments. We consider an elliptical fuselage cross-section and develop a theoretical analysis to optimise its skin thickness in function of the polar angle, pressure loading and eccentricity. A comparison of the proposed elliptical fuselage with a circular one for different scenarios is developed. With a lightweight core and optimised variable skin thickness above and below the core, the elliptical shape with an unsymmetrical sandwich construction can have a smaller volume of skins than the circular one. This implies potential weight gains to be made with an elliptical fuselage.

1. Introduction

Pressurised vessels such as aircraft fuselages are commonly designed so that they have a circular or lobed cross-section. Circular cross-sections are ideal to resist pressurisation since the internal pressure can be balanced by purely tensile loading of the membrane. However, in the design of an aircraft fuselage, other considerations must be taken into account: maximised internal space for passenger comfort, sufficient clearance above the passengers with window seats, below-floor space for systems wiring, etc. These considerations may lead engineers to consider other cross-section geometries. For example, elliptical fuselage cross-sections offer more flexibility in designing comfortable passenger compartments with increased seating, optimised use of interior space, and potential improvements in aerodynamic performances by approaching a blended wing body configuration. However, any deviation from a circular cross-section design gives rise to bending moments in the fuselage panels upon pressurisation [1].

In a fuselage of elliptical cross-section, pressurisation induces a distribution of hoop force, shear force and bending moment along the perimeter [1, 2, 3, 4] in comparison with a sole constant hoop force in a circle. Sandwich constructions, i.e., thin composite skins separated by a lightweight core, are efficient at bearing bending moments [5, 6, 7]. They also offer advantages for future aircrafts as their characteristics can be tailored in terms of structural properties as well as noise and thermal insulation [6, 8].

In the present paper, we consider an elliptical fuselage cross-section and develop a theoretical analysis to optimise its skin thickness. We optimise the skin thickness in function of the polar angle, pressure loading and eccentricity of the ellipse. An analytical thin shell theory is applied [2] to compute the loading due to pressurisation of the shell. A comparison of the proposed elliptical fuselage with a circular one for different scenarios is developed. We systematically analyse the effect of elliptical eccentricity on skin thickness and skin volume variations. The possibility of having an elliptical cross-section that would provide more space for the passengers while limiting the weight penalty compared to a circle is discussed.

2. Methodology

We consider a non-reinforced thin shell fuselage of elliptical cross-section defined by its large semi-axis a, and small semi-axis b. The ellipse eccentricity is defined as the ratio b/a. We compare the elliptical cross-section to a circular one defined by a constant radius r. The radius of the elliptical cross-section with respect to θ is defined by a polar equation as follows

$$r_{ellipse}(\theta) = \frac{ab}{\sqrt{b^2\cos(\theta)^2 + a^2\sin(\theta)^2}}. \tag{1}$$

The shell thickness is sized to carry the load due to internal pressurisation of the fuselage. In order to compare the elliptical cross-section to a circular one, two design scenarios are defined (Fig. 1): same inner area, and same enclosed rectangle. The ellipse and circle of Fig. 1 (*a*) have the *same area*, i.e., $\pi ab = \pi r^2$. Thus, the ratio of the large semi-axis of the ellipse to the corresponding circle radius is related to the eccentricity of the ellipse

$$\frac{a}{r} = \sqrt{\frac{a}{b}}. \tag{2}$$

The ellipse and circle of Fig. 1 (b) have the *same enclosed rectangle* of dimensions L by W. The smallest circle enclosing the rectangle obeys the relation $L^2 + W^2 = 4r^2$. An infinity of ellipses enclose the same rectangle, they obey the relation $b^2L^2 + a^2W^2 = 4a^2b^2$. Among them, the one ellipse with the minimal perimeter also obeys the relation $b^2L = a^2W$. The ratio of the large semi-axis of the ellipse with minimal perimeter to the corresponding circle radius can be related to the aspect ratio of the enclosed rectangle

$$\frac{a}{r} = \sqrt{\frac{1 + W/L}{1 + (W/L)^2}}. \tag{3}$$

Assuming a small shell thickness compared to the shell radius (or to the shell semi minor axis), we can evaluate the loads due to an internal pressurisation P. An elliptical shell at equilibrium carries a tangential force per unit width N varying with θ [1]:

$$N = P\sqrt{a^2\cos(\theta)^2 + b^2\sin(\theta)^2}, \tag{4}$$

The pressurisation also gives rise to a distribution of bending moment per unit width. Upon integration of the equilibrium equations, the following distribution is obtained [1]:

$$M = \frac{Pa^2}{2}\left(D - \frac{P^2b^2}{N^2}\right), \tag{5}$$

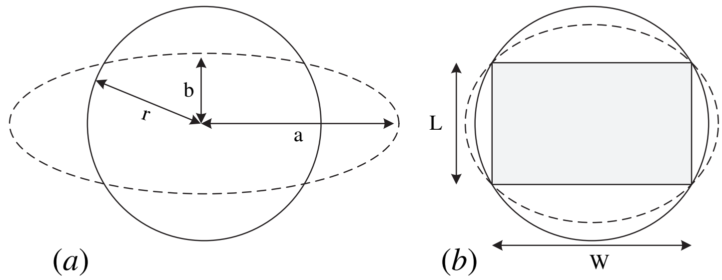

Figure 1. Schematics of the two design scenarios: (*a*) the circle and the ellipse have the *same area*; (*b*) the circle and the ellipse enclose the *same rectangle* of dimensions $W \times L$ and have minimal perimeter.

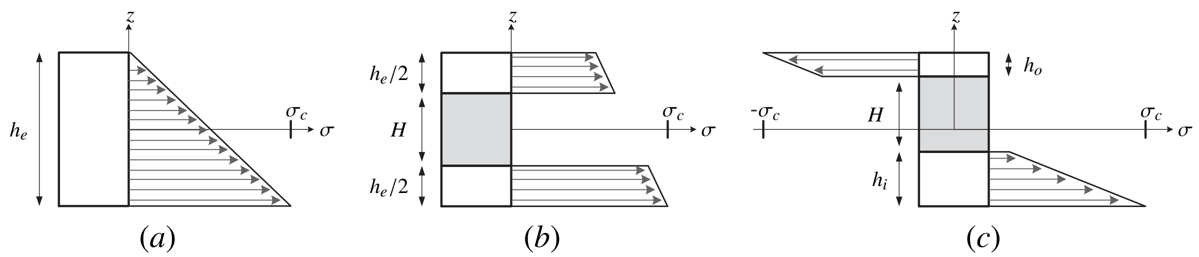

Figure 2. Schematic of the stress through the thickness for three elliptical shell constructions: (*a*) monolithic material; (*b*) symmetrical sandwich; (*c*) unsymmetrical sandwich.

where $D = 0.6592281\,(b/a - 0.26)^{2.14} + 0.653908$ is a fit of the integration constant obtained numerically for various values of the eccentricity [1]. To design the elliptical shell, three composite constructions are considered: monolithic, symmetrical sandwich, and unsymmetrical sandwich (Fig. 2). In the monolithic construction, the skin thickness of the ellipse h_e varies with the polar angle θ and is sized to bear all circumferential loads. For both sandwich composite constructions, a core of thickness H is considered. In the case of the symmetrical sandwich construction, the skin thickness is $h_e(\theta)/2$ on each side of the core. For the unsymmetrical sandwich construction, the outer skin thickness is $h_o(\theta)$ and the inner skin thickness is $h_i(\theta)$. The total skin thickness in this last case is $h_e(\theta) = h_o(\theta) + h_i(\theta)$. For the sake of comparison, in both design criteria and all three constructions, the thickness of the circular cross section is defined as h_c. The tangential stress in the shell varies through its thickness and is due to the tangential force and the bending moment [2]:

$$\sigma_{ellipse}(\theta, z) = \frac{N}{h_e} + \frac{z\,M}{I}, \tag{6}$$

where $I(\theta)$ is the second area moment of the shell per unit length. The coordinate z is the distance between the neutral plane of the shell and the point where the stress is evaluated. In a circle, the eccentricity is null, the bending moment disappears and the hoop stress is assumed constant through the thickness:

$$\sigma_{circle} = \frac{Pr}{h_c}. \tag{7}$$

In all three constructions considered, the thickness is sized so that the maximum stress in the ellipse is equal to the stress in the circle for the same loading. This criteria is enforced for all values of θ, i.e.,

$$\max \sigma_{ellipse}(z) = \sigma_{circle} \ \forall\ \theta. \tag{8}$$

The tangential stress distribution through the thickness of the elliptical cross-section shell is shown in Fig. 2 for all three construction cases. In the monolithic construction, Eq. 8 is satisfied at the outer or inner edge of the shell such that the maximum stress at $z = \pm h_e/2$ is equal to the circle's hoop stress for the same applied pressure P. For the monolithic construction, $I = h_e^3/12$. Equating Eqs 6 and 7 at $z = \pm h_e/2$ and taking the maximum value of thickness, we find

$$\frac{h_e}{h_c} = \frac{N}{2Pr} + \sqrt{\frac{1}{4}\left(\frac{N}{Pr}\right)^2 + \left|\frac{6M}{Prh_c}\right|}. \tag{9}$$

Sandwich constructions offer significant bending rigidity through an increase of the second area moment of the shell due to a separation of the skins by a lightweight core. It is assumed that the core has negligible weight and the hoop stress due to the pressure is bore by the skins only. For a symmetrical sandwich, the second area moment is

$$I = \frac{h_e}{4}\left(\frac{h_e^2}{12} + \left(H + \frac{h_e}{2}\right)^2\right). \tag{10}$$

Substituting this expression in the stress formulation Eq. 6 and comparing it to the circle stress formulation Eq. 7 via the criteria equation 8, we obtain a third order polynomial of h_e/h_c:

$$\frac{1}{3}\left(\frac{h_e}{h_c}\right)^3 + \left(\frac{H}{h_c} - \frac{N}{3Pr}\right)\left(\frac{h_e}{h_c}\right)^2 + \left(\frac{H^2}{h_c^2} - \frac{HN}{Prh_c} - \left|\frac{2M}{Prh_c}\right|\right)\left(\frac{h_e}{h_c}\right) + \frac{H}{Prh_c^2}\left(-NH - |2M|\right) = 0. \tag{11}$$

The polynomial Eq. 11 has three roots. It can be shown that two of those roots are imaginary and one is real. We disregard the imaginary roots and obtain a formulation for h_e/h_c in function of θ and b/a by solving Eq. 11.

A symmetrical sandwich construction provides an important advantage over a monolithic construction. However, a symmetrical sandwich is not fully optimised as the maximum stress equivalent to the circle stress is found at only one location through the thickness, i.e., either at the outer or inner edge of the sandwich. To further reduce the weight of the skins, we finally consider an unsymmetrical sandwich. For this last construction, we impose that the maximum stress is reached in *both* the outer and inner skins, i.e., we impose that Eq. 8 is enforced at both the outer and inner edges. For an unsymmetrical construction, the second area moment can be formulated as:

$$I = \frac{h_o^3}{12} + h_o z_o^2 + \frac{h_i^2}{12} + h_i z_i^2, \tag{12}$$

where z_o and z_i are the distances from neutral plane of the composite to the centre of the outer and inner skins respectively. Depending on the thickness of each skin, the neutral plane will not necessarily be at the center of the core. The distances z_o and z_i can be evaluated by finding the neutral plane

$$z_o = \frac{Hh_i}{h_o + h_i} + \frac{h_i}{2}, \quad z_i = \frac{Hh_o}{h_o + h_i} + \frac{h_o}{2}. \tag{13}$$

Thus, for an unsymmetrical sandwich construction, we find one non-linear equation for each skin that must be respected for the same polar angle:

$$\frac{h_o}{h_c} = \frac{2NI}{2PrI - Mh_c\left(2z_o - h_o\right)} - \frac{h_i}{h_c}, \tag{14}$$

$$\frac{h_i}{h_c} = \frac{2NI}{2PrI + Mh_c\left(2z_i - h_i\right)} - \frac{h_o}{h_c}. \tag{15}$$

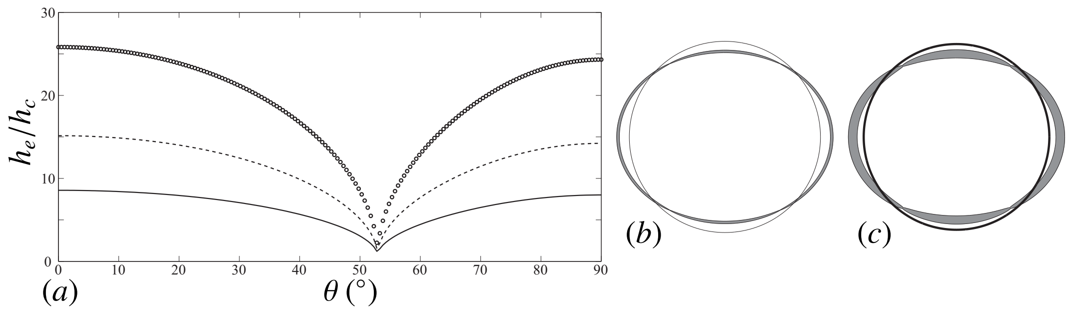

Figure 3. Same area scenario and monolithic construction with eccentricity $b/a = 0.8$. (a) Thickness ratio between the elliptical and circular shape h_e/h_c for circular shell aspect ratios r/h_c =: 90 (—); 300 (--) ; and 900 (∘∘). Scaled drawings comparing the circular (black) and elliptical (grey) sections for $r/h_c = 900$ (b); and $r/h_c = 90$ (c).

Those equations are implicit and coupled, indeed, I, z_o and z_i depend on h_o and h_i. The solver function *fsolve* in MATLAB with the iterative algorithm *Levenberg-Marquardt* is used to solve the system of equations. A continuation method is used: the thickness values solving the equations for a given θ angle are used as initial guesses to compute the thickness values for the next angle.

For all three constructions, once the skin thickness distribution about θ is available, it is numerically integrated to obtain the total volume of skin material which must be used in the elliptical cross-section in comparison to the circular one

$$\frac{V_e}{V_c} = \frac{2}{\pi r} \int_0^{\pi/2} \frac{h_e}{h_c} r_e \, d\theta. \tag{16}$$

3. Results

We first analyse the results of the same area design scenario. For the monolithic construction, the variation with θ of the elliptical shell thickness in comparison with the circular shell thickness is shown in Fig. 3 (a). The results are presented for three different comparative circular shell aspect ratios r/h_c: a thin (∘∘), a medium (--) and a thick shell (—). The relative thickness h_e/h_c is larger for thinner circular shells. For a monolithic construction, to limit the stress in the elliptical shell to the level of that in a thin circular shell ($r/h_c = 900$, (∘∘)), the elliptical shell must have a thickness which varies between 1 and 26 times the thickness of the circular shell. In comparison to a thick circular shell ($r/h_c = 90$, —), the elliptical shell must vary between 1 and 8 times the thickness of the circular shell. Considering that even for a thick shell, the thickness ratio is almost of order 10 for most of the perimeter; the elliptical cross-section is very inefficient. Interestingly, in Fig. 3 all three curves have a discontinuity and a minimum at $\theta \simeq 53°$. At this angle, the bending moment per unit width is null. The value of 53° is solely dictated by the eccentricity of the shell. The cross-sections of the elliptical shells compared to circular ones of same area are plotted in Fig. 3 (b-c) with their thickness drawn to scale. The increase in wall thickness for the elliptical shells is obvious. Because of the inefficiency of monolithic thin shells to carry bending moments, we do not consider monolithic constructions any further.

In order to increase the bending stiffness by significantly increasing the second area moment with the separation of the skins by a lightweight core, sandwich constructions are analysed. We

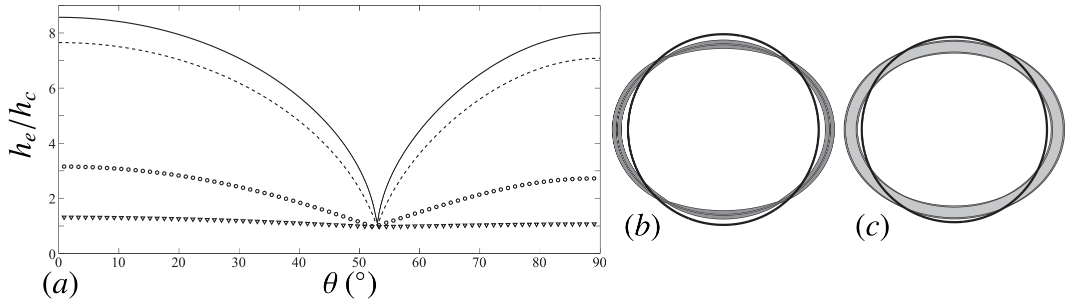

Figure 4. Same area scenario and symmetrical sandwich construction with eccentricity $b/a = 0.8$ and circular shell aspect ratios $r/h_c = 90$. (*a*) Thickness ratio between the elliptical and circular shape h_e/h_c for ratio of the core thickness and the circle thickness H/h_c =: 0 (—); 1 (--); 10 (○○); 100 (▽▽). Scaled drawings comparing the circular (black) and elliptical (dark grey for the skins, light grey for the core) sections for $H/h_c = 1$ (*b*); and $H/h_c = 10$ (*c*).

assume that the stress due to the pressure is bore by the skins only and the core has negligible weight compared to the skins. For the symmetrical construction, the variation with θ of the relative thickness of the elliptical shell with respect to the circular shell is shown in Fig. 4 (a). Note that the *y*-scale in Fig. 4 is different than in Fig. 3. The results in Fig. 4 (a) are presented for four different dimensionless core thickness H/h_c: without core (—, same curve as $r/h_c = 90$ in Fig. 3), with a thin (--), a medium (○○) and a thick core (▽▽). For a thin core (--), to limit the stress in the elliptical shell to the level of that in a circular shell, the elliptical shell must be up to 7.5 times the thickness of the circular shell. With a thick core (▽▽), the elliptical shell must have a thickness ratio which varies between 1 and 1.5 that of the circle. Even for this thick core, the skins of the elliptical cross-section must be thicker than for the circular shell. Similarly as for the monolithic construction (Fig. 3), all four curves have a discontinuity and a minimum at $\theta \simeq 53°$. As the maximum stress equivalent to the circle hoop stress is found at the outer *or* inner edge of the sandwich, the symmetrical sandwich construction is not optimal. Therefore we turn our attention to an unsymmetrical sandwich construction.

For the unsymmetrical sandwich construction and the same area scenario, Eqs. 14 and 15 are solved using an optimising program applying the same absolute stress at *both* the outer and inner edges. The variation of the relative thickness is shown in Fig. 5 (a). The results are presented for two different core thickness H/h_c: a medium (○○) and a thick core (—). The relative thickness h_e/h_c is larger for a thinner core. For an unsymmetrical sandwich construction with a thick core ($H/h_c = 100$), the elliptical shell has a skin thickness which varies between 0.92 and 1.14 times that of the circular shell. For a core of medium thickness ($H/h_c = 10$, ○○), the total thickness of the skins of the elliptical shell must be up to 1.4 times the thickness of the circular shell. To better evaluate the cost or benefit of an elliptical fuselage, the volume of the skins of the ellipse normalised by that of the circle is evaluated using Eq. 16. This normalised volume is shown in Fig. 5 (b) as a contour plot in function of the core thickness and the ellipse eccentricity. In the range of parameters tested, the volume of skin of the elliptical cross-section is always higher than that of the circular cross-section. However, for large values of H/h_c and small values of b/a there is a contour line of $V_e/V_c = 1.01$ beyond which the elliptical shape is less than 1% heavier than the circular shape. If the elliptical cross section design brings other benefits, a small weight gain could be acceptable.

The results obtained for the unsymmetrical construction with the same enclosed rectangle sce-

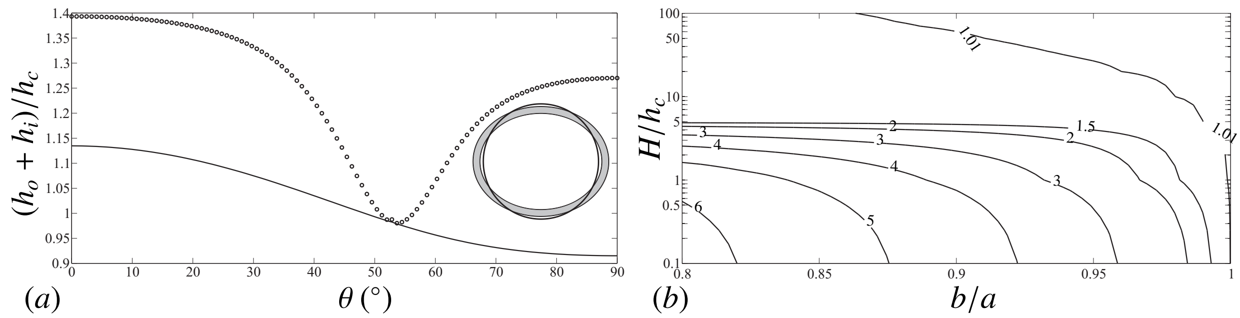

Figure 5. Same area scenario and unsymmetrical construction with eccentricity $b/a = 0.8$ and circular shell aspect ratios $r/hc = 90$. (*a*) Thickness ratio between the elliptical and circular shape h_e/h_c for ratio of the core thickness and the circle thickness H/h_c =: 10 (∘∘); 100 (—) with scaled drawing comparing the circular (black) and elliptical (dark grey for the skins, light grey for the core) sections for $H/h_c = 10$. (*b*) Isolines of the elliptical shell skins volume normalised by the circular shell skins volume (V_e/V_c).

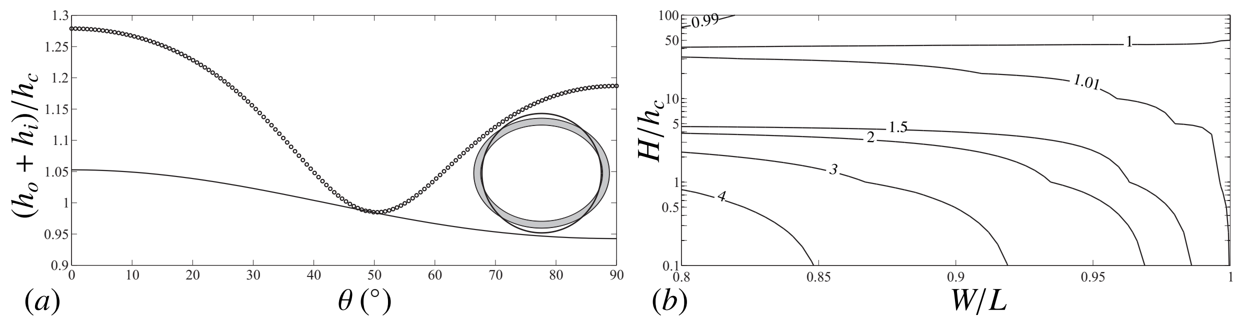

Figure 6. Same enclosed rectangle scenario and unsymmetrical construction with rectangle dimensions $W/L = 0.8$ and circular shell aspect ratios $r/hc = 95$. (*a*) Thickness ratio between the elliptical and circular shape h_e/h_c for ratio of the core thickness and the circle thickness H/h_c =: 10 (∘∘); 100 (—) with scaled drawing comparing the circular (black) and elliptical (dark grey for the skins, light grey for the core) sections for $H/h_c = 10$. (*b*) Isolines of the elliptical shell skins volume normalised by the circular shell skins volume (V_e/V_c).

nario are shown in Fig. 6. In Fig. 6 (a), for a medium core ($H/h_c = 10$, ∘∘), the total thickness of the skins of the elliptical shell must be up to 1.27 times the thickness of the circular shell. For a thick core ($H/h_c = 100$, —), the elliptical shell must have a total skin thickness which varies between 0.94 and 1.05 times that of the circular shell. In Fig. 6 (b), a contour plot of the normalised volume (Eq. 16) of the shell skins is shown in function of the core thickness and the aspect ratio of the enclosed rectangle. Recall that for a given enclosed rectangle aspect ratio, the ellipse considered is the one with the smallest perimeter. In the parameter range investigated, an isoline of $V_e/V_c = 1$ exist. For W/L varying from 0.8 to 1, the volume of the skins in an ellipse are the same as in a circle for a core thickness H/h_c varying between 40 and 50. In sizing a fuselage to accommodate a cabin with an imposed rectangular shape, an elliptical cross section can be advantageous if a fairly thick ($H/h_c > 50$) core in an unsymmetrical sandwich is considered.

4. Conclusion

This paper shows that an elliptical cross-section with a monolithic construction is not interesting when comparing the thickness the shell with a circular one having the same inner area. Inserting a core between two identical skins and creating a symmetrical sandwich is more interesting but the structure of an elliptical fuselage cross-section cannot be lighter than the circular one for the design scenarios investigated here. It is shown that replacing a circular cross-section by an elliptical one with an unsymmetrical sandwich construction and the same enclosed rectangle offers advantages for aircraft design. Indeed, with a lightweight core and variable skin thickness above and below the core, the elliptical shape can be optimised to have a smaller volume of skins than the circular one. This implies potential weight gains to be made with an elliptical fuselage.

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