DYNAMIC CRACK ANALYSIS IN ANISOTROPIC FUNCTIONALLY GRADED MATERIALS BY A TIME-DOMAIN BEM

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Abstract
In this paper, the dynamic crack analysis in two-dimensional (2D) anisotropic and functionally graded composites is presented. For this purpose, a time-domain boundary element method (BEM) is developed. The present BEM uses the collocation method for the spatial discretization of the time-domain boundary integral equations (BIEs), while the convolution quadrature of Lubich is applied for temporal discretization. The Laplace-domain fundamental solutions for homogenous anisotropic materials are implemented. Special regularization techniques are used to deal with the singular boundary integrals and the resulting domain integrals. An explicit time-stepping scheme is obtained to compute the unknown boundary data. Numerical examples are presented to show the influences of the material gradation and the transient dynamic loadings on the dynamic stress intensity factors.

1. Introduction

Functionally graded materials (FGMs) offer many possibilities in advanced engineering structures. The main characteristic of these materials is a continuously change of the material properties. This can be used to design FGMs with the most beneficial mechanical and thermal material behavior. Another important aspect is that interfaces and stress singularities, as in conventional laminates, are avoided. An essential task in the application and the safety of FGM structures is the fracture and damage analysis. Since such composites are often subjected to severe loading conditions such as impact or time-dependent loading, the dynamic crack analysis is of special interest. The analysis of FGMs is mathematically complex and analytical or semi-analytical solutions are possible only for very simple geometry and loading conditions. Therefore, efficient numerical methods are necessary to solve more general problems. Beside the well known finite element method (FEM) the meshless local Petrov-Galerkin method (MLPG) has been developed for anisotropic elasticity for functionally graded materials by Sladek et al. [8]. The boundary element method (BEM) is a very attractive and accurate numerical tool for crack problems in homogeneous materials but its application to non-homogeneous material is rather limited. This is due to the fact that the corresponding required fundamental solutions for FGMs are either mathematically very complicated or not available [1]. Gao et al. [4] have presented a collocation BEM for plane isotropic FGMs under static loading. A Laplace-domain BEM for the transient coupled thermoelastic crack analysis in 2D functionally graded isotropic materials and structures have been developed by Ekhlakov et al. [2]. In these works, to avoid the use of the special fundamental solutions for non-homogeneous material the derived boundary integral equations contain domain integrals which makes the application of the fundamental solutions for homogeneous materials possible. By using the radial integration method [3] the domain integrals can be transformed into equivalent boundary integrals. Although the TDBEM has been successful applied for homogeneous anisotropic and linear elastic solids [5, 9] its extension to anisotropic functionally graded materials is to the authors best knowledge an open task.

2. Problem statement

Let us consider a continuously non-homogeneous anisotropic and linear elastic solid containing a finite and stationary crack. Without applied body forces, the cracked solid satisfies the equations of motion

$$\sigma_{ij,j}(\mathbf{x},t) = \rho(\mathbf{x})\ddot{u}_i(\mathbf{x},t), \tag{1}$$

the constitutive equations

$$\sigma_{ij}(\mathbf{x},t) = c_{ijkl}(\mathbf{x})u_{k,l}(\mathbf{x},t) \tag{2}$$

with

$$c_{ijkl}(\mathbf{x}) = \Theta(\mathbf{x})c^0_{ijkl}, \tag{3}$$

the initial conditions for $t \leq 0$

$$u_i(\mathbf{x},0) = \dot{u}_i(\mathbf{x},0) = 0, \tag{4}$$

and the boundary conditions

$$u_i(\mathbf{x},t) = \bar{u}_i(\mathbf{x},t), \qquad \mathbf{x} \in \Gamma_u, \tag{5}$$

$$t_i(\mathbf{x},t) = \bar{t}_i(\mathbf{x},t), \qquad \mathbf{x} \in \Gamma_t, \tag{6}$$

where u_i, σ_{ij} and t_i denote the displacement, the stress and the traction components, ρ is the mass density, c^0_{ijkl} is the elasticity tensor, Θ defines the spatial variation of the material properties, Γ_u is the external boundary where the displacements u_i are prescribed, Γ_t represents the external boundary where the tractions t_i are given. The crack is considered as traction free

$$t_i(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \Gamma_c, \tag{7}$$

where Γ_c=Γ_{c^+} + Γ_{c^-} with Γ_{c^+} and Γ_{c^-} denoting the upper and the lower crack-faces and the traction vector $t_i(\mathbf{x},t)$ is defined by

$$t_i(\mathbf{x},t) = \sigma_{ij}(\mathbf{x},t)e_j(\mathbf{x}). \tag{8}$$

In Eq. (8) $e_j(\mathbf{x})$ denotes the outward unit normal vector. Throughout the paper, a comma after a quantity designates spatial derivatives, while superscript dots stand for temporal derivatives of the quantity, a Roman suffix takes the values of 1 and 2.

3. Time-domain BIEs and fundamental solutions

In order to solve the corresponding initial boundary value problem with the BEM, it is formulated as boundary integral equations (BIEs). Following the procedure given in Gao et al. [4] the BIEs can be written as

$$c_{ij}\tilde{u}_j(\mathbf{x},t) = \int_\Gamma \left[u^G_{ij}(\mathbf{x},\mathbf{y},t) * t_j(\mathbf{y},t) - t^G_{ij}(\mathbf{x},\mathbf{y},t) * \tilde{u}_j(\mathbf{y},t)\right]\mathrm{d}\Gamma_y + \int_\Omega h^G_{ij}(\mathbf{x},\mathbf{y},t) * \tilde{u}_j(\mathbf{y},t)\mathrm{d}\Omega_y, \quad (9)$$

where $u^G_{ij}(\mathbf{x},\mathbf{y},t)$, $t^G_{ij}(\mathbf{x},\mathbf{y},t)$ and $h^G_{ij}(\mathbf{x},\mathbf{y},t)$ are the fundamental solutions, $\tilde{u}_j(\mathbf{x},t) = \Theta(\mathbf{x})u_j(\mathbf{x},t)$ are the normalized displacements and an asterisk "$*$" denotes the Riemann convolution. The free term c_{ij} is defined by $c_{ij} = \frac{1}{2}\delta_{ij}$ for the source point $\mathbf{x}$ on the smooth boundary Γ and $c_{ij} = \delta_{ij}$ for $\mathbf{x}$ inside the domain Ω, with δ_{ij} being the Kronecker delta.
In this work, the Riemann convolution integral is approximated by the convolution quadrature of Lubich [6, 7] which requires the Laplace-domain fundamental solutions instead of the time-domain fundamental solutions. The Laplace-domain fundamental solutions for homogeneous, anisotropic and linear elastic solids can be represented in the 2D case by a line integral over the unit-circle as

$$u^G_{ij}(\mathbf{x},\mathbf{y},p) = \frac{1}{8\pi^2}\int_{|\mathbf{n}|=1}\sum_{m=1}^{M}\frac{P^m_{ij}}{\rho c_m^2}\Psi\left(\frac{p}{c_m}, \left|\mathbf{n}\cdot(\mathbf{y}-\mathbf{x})\right|\right)\mathrm{d}\mathbf{n}, \quad (10)$$

$$\Psi(\xi) = \left[\mathrm{e}^{\xi}\mathrm{Ei}(-\xi) + \mathrm{e}^{-\xi}\mathrm{Ei}(\xi)\right], \qquad \xi = \left(\frac{p}{c_m}\left|\mathbf{n}\cdot(\mathbf{y}-\mathbf{x})\right|\right). \quad (11)$$

In Eqs. (10) and (11), p, Ei, $\mathbf{n}$, c_m and P^m_{ij} are the Laplace parameter, the exponential integral, the wave propagation vector, the phase velocities of the elastic waves and the projection operator as given in [5, 9]. The fundamental solutions can be divided into a static part and a dynamic part

$$u^G_{ij}(\mathbf{x},\mathbf{y},p) = u^S_{ij}(\mathbf{x},\mathbf{y}) + u^D_{ij}(\mathbf{x},\mathbf{y},p). \quad (12)$$

The static part can be simplified to a explicit form and contains the singularities while the dynamic part is regular and still defined by a line integral over the unit-circle.

4. Numerical solution procedure

For the spatial discretization the boundary Γ is approximated by quadratic elements and internal nodes are defined inside the domain Ω. Quarter-point elements are used adjacent the crack-tips to describe the local behavior of the crack opening displacements (CODs) at the crack-tips properly. In order to get a solvable system of linear algebraic equations the BIEs (9) are written for all boundary nodes and internal nodes. The boundary integrals in Eq. (9) are computed numerically. A regularization technique based on a suitable change of variable is applied in this paper, as shown by García-Sánchez et al. [5]. It should be mentioned that fully analytical integration is possible for straight elements. The line-integrals over the unit-circle in the Laplace-domain elastodynamic fundamental solutions have to be computed numerically, which can be done by standard Gaussian quadrature formula. The domain integrals in the BIEs (9) can be computed directly by cell integration over Ω. In this case an additional mesh has to be defined in the domain and the BEM loses its basic idea and main advantage. To avoid an additional mesh the radial integration method (RIM) [3] is applied to transform the domain integrals in equivalent boundary integrals in the present work. As suggested in [4, 2] the normalized displacements are approximated by a combination of radial basis functions and the linear polynomials. A fourth order spline-type radial basis function is used. The integration with respect to the distance r in the RIM can be computed analytically which enhance the efficiency of the presented BEM. The boundary integration and the integration over the unit-circle can be computed numerically by the standard Gaussian quadrature formula.

After spatial and temporal discretization the following explicit time-stepping scheme is obtained to compute the discrete boundary data step by step

$$\mathbf{g}^K = (\mathbf{A}^1)^{-1}\left[\mathbf{B}^1\mathbf{h}^K + \sum_{k=1}^{K-1}\left(\mathbf{B}^{K-k+1}\mathbf{h}^k - \mathbf{A}^{K-k+1}\mathbf{g}^k\right)\right]. \tag{13}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the time-domain system matrices, $\mathbf{g}$ is the vectors containing the unknown boundary data and normalized internal displacements and $\mathbf{h}$ is the vector of the prescribed boundary data.

5. Dynamic stress intensity factors

The crack-tip field for a crack inside a functionally graded non-homogenous material shows the well known square-root behavior as for the crack in a homogenous material. Since quarter-point crack-tip elements are implemented in the present time-domain BEM to describe this local behavior of the CODs near the crack-tips properly, the dynamic stress intensity factors (SIF) can be obtained directly without special techniques from the numerically computed CODs at the closest nodes to the crack-tips by

$$\left\{\begin{array}{c} K_I(t) \\ K_{II}(t) \end{array}\right\} = \frac{1}{4\Delta}\sqrt{\frac{2\pi}{l}}\mathbf{H}\left\{\begin{array}{c} \Delta u_1(l,t) \\ \Delta u_2(l,t) \end{array}\right\}. \tag{14}$$

Here, Δ and $\mathbf{H}$ are defined in [9] and l is the distance between the crack-tip and the closest node.

6. Numerical examples

In the following, a numerical example is presented and discussed to show the effects of material gradation and the dynamic loading on the dynamic intensity factors (IFs). For convenience, the following normalized dynamic stress intensity factor are introduced

$$K_I^*(t) = K_I(t)/\sigma_0\sqrt{\pi a}, \qquad K_{II}^*(t) = K_{II}(t)/\sigma_0\sqrt{\pi a}, \tag{15}$$

where σ_0 is the loading amplitude, and a is the half-length of an internal crack or the length of an edge-crack.

As example let us consider a functionally graded rectangular plate with a central crack subjected to an impact loading of the form $\sigma(t) = \sigma_0 \mathrm{H}(t)$ normal to the crack-face. The geometry of the cracked plate as shown in Figure 1 is determined by $h = 20.0mm$, $2w = h$ and $2a = 4.8mm$.

An orthotropic Graphite-Epoxy composite is investigated, which has a mass density of $\rho =$

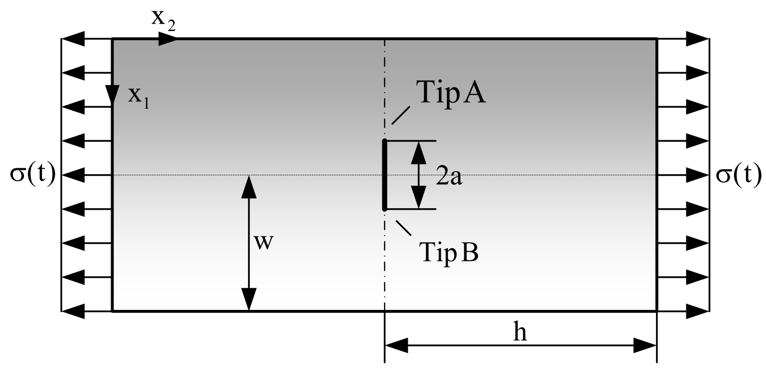

Figure 1. A functionally graded rectangular plate with a central crack under impact tensile loading

$1600 kg/m^3$ and on the top of the plate the following elastic constants

$$C_{ij}^0 = \begin{pmatrix} 155.43 & 3.72 & 0 \\ & 16.34 & 0 \\ sym & & 7.48 \end{pmatrix} GPa. \tag{16}$$

An linear variation in x_1- direction is assumed where the material parameters on the bottom surface of the plate are doubled with respect to ones on the top. The mass density is considered as constant and plane strain condition is assumed. The external boundary is discretized by using a uniform mesh with an element-length of $2.0mm$ and the crack is divided into 6 elements. Inside the domain uniform distributed nodes with the distance of $2.0mm$ are used. A time-step of $\Delta t = 0.22\mu s$ is chosen. Since the cracked plate is symmetric with respect to the crack plane only the right half is used in the numerical computation. The numerical results of the present BEM for the Tip A, the Tip B and for the homogenous material are shown in Figure 2.

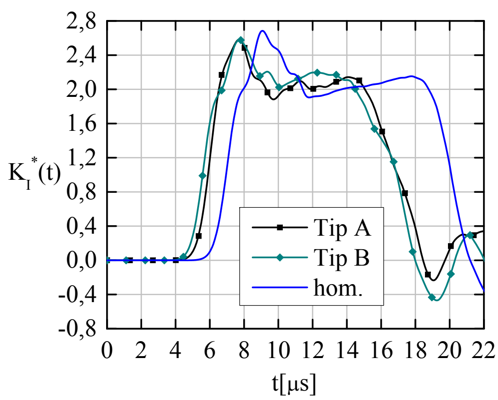

Figure 2. Normalized dynamic $K(t)$-factors of both crack-tips and for the homogenous case

The global behavior of the dynamic mode-I stress intensity factors are similar without significant differences. From Figure 2 it can be seen that the dynamic stress intensity factors are zero until the elastic waves reach the crack. Immediately after the waves reach the crack, the dynamic stress intensity factors increase rapidly with the time and after reaching a peak they show a complex variation. The wave velocities of the FGM increase in the x_1-direction. As a consequence, the dynamic mode-I stress intensity factor of the Tip B increases at a earlier time instant than which of the Tip A. In the investigated linear variation of the elastic constants the difference between Tip A and Tip B are small. The mode-II stress intensity factors vanish for the applied loading normal to the crack-faces.

Summary

A time-domain boundary element method for the dynamic crack analysis in two-dimensional (2D) anisotropic and functionally graded composites is presented. The BEM uses the collocation method for the spatial discretization and the convolution quadrature for temporal discretization. The RIM is used to transform the domain integrals into equivalent boundary integrals. First numerical examples show the influences of the material gradation on the dynamic stress intensity factors and indicate the suitability of the BEM for this class of materials.

Acknowledgements

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