FLEXURAL BEHAVIOUR OF SANDWICH PANELS WITH GRADED CORE: A PIECEWISE EXPONENTIAL MODEL

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Abstract

In this paper three-dimensional elastic deformation of rectangular sandwich panels with graded core subjected to transverse loading is investigated using a recently developed three-dimensional elasticity solution which has been extended to produce a piecewise exponential model. This model can be applied to any functionally graded plate or sandwich panel with graded core as long as the through thickness stiffness variation can be described by a smooth function. The new approach is fully validated through both comparison with results from the literature and a three-dimensional finite element method which employs user-implemented graded finite elements. As an example the new approach is applied to analysis of sandwich panels with power law variation in stiffness properties of the core and a study is carried out to examine the effect of varying the power law index on panel's response.

1. Introduction

Sandwich panels or plates are structures that can be used in place of standard plates or panels, offering weight savings whilst remaining strong and stiff. They are used in a variety of engineering applications particularly in the aerospace industry. Due to the mismatch in stiffness properties between the face sheets and the core, sandwich panels are susceptible to delamination, caused by high interfacial stresses, especially under localised loading [1] or at high service temperatures [2]. One effective method of minimising the large interfacial shear stresses is to make use of a functionally graded material for the panel core. Functionally graded materials are a type of heterogeneous composite materials exhibiting gradual variation in microstructure and composition of the two constituent materials from one surface of the material to the other, resulting in properties which vary continuously across the material. A number of researchers have developed analytical solutions for sandwich panels with graded core. Anderson [3] developed a 3D elasticity solution for a sandwich panel with orthotropic face sheets and an isotropic functionally graded core subjected to transverse loading by a rigid sphere. Kashtalyan and Menshykova [4] recently developed a three dimensional elasticity solution for sandwich panels with functionally graded core whose shear moduli vary exponentially through the thickness of the core. Apetre et al [5] investigated several available sandwich beam theories for their suitability of application to sandwich plates with functionally graded core. All of the three-dimensional results above assume a through thickness exponential variation in stiffness properties. However, there may be benefits in utilising cores with other, non-exponential distributions in stiffness properties and so methods must be sought in order to model them. In addition, after fabrication, real life functionally graded materials may not have exponential variation in stiffness properties, so it is important that other distributions of stiffness can be modelled. Several researchers have developed solutions for functionally graded plates with power law variation in stiffness properties. This power law variation has the advantage that the stiffness at any point through the thickness is directly linked to volume fraction at that point and the stiffness's of the two constituent materials in their homogeneous form. Zenkour [6] presented a simplified generalised shear deformation theory for a functionally graded rectangular plate under uniform and sinusoidal loads. The work utilises a FGM comprising a ceramic and metal, with through thickness power law variation in stiffness, directly related to the volume fraction of the constituents. Stresses and displacements are studied and the gradient is shown to play an important role with the results for the graded material lying in between the pure ceramic and pure metal. Navazi and Haddadpour [7] and GhannadPour and Alinia [8] both consider the bending of a functionally graded plate, once more with power law variation in stiffness properties directly linked with the volume fraction. Von-Karman large deflection theory is then used in both cases to simply the differential equations and a comparative study into the stresses and displacements in a graded and homogenous plate is carried out. Both papers report that the functionally graded plate exhibits significant non-linear behaviour, which is absent in the analysis of the homogenous plate. A functionally graded plate utilising the Mori-Tanaka scheme and power law variation in material properties was studied by [9]. In this work First order shear deformation theory is used to confirm that functionally graded plates behave differently under an upward or downward transverse load. Yaghoobi and Fereidoon [10] consider the effect of varying the power law index the position of the neutral surface and deflections of a functionally graded beam. They conclude that in a metal/ceramic functionally graded beam, the neutral surface shifts towards the ceramic rich surface and as the power law exponent is increased, the maximum deflection of the beam also increases. Tornabene [11] uses first order shear deformation theory to model the dynamic behaviour of moderately thick functionally graded shells. In this case the influence of a four parameter power law stiffness variation is examined and is found to have a large effect on the dynamic vibration behaviour. Alshorbagy et al [12] use a finite element method to explore the vibration behaviour of a functionally graded beam. In the current work, the recently developed 3D elasticity solution for sandwich panels with exponential variation of the Young's modulus through the thickness [4] is expanded to produce a piecewise exponential model. This model can be applied to any functionally graded plates or sandwich panels with graded core as long as the through thickness stiffness variation can be described by a smooth function. This new method is fully validated through both comparison with results from the literature and a finite element study. As an example the method is then applied to power law variation in stiffness properties and a study is carried out to examine the effect of varying the power law index.

2. Problem formulation

A sandwich panel (Fig. 1) of length a, width b and total thickness $h_0 = 2h$ is referred to a Cartesian co-ordinate system x_1, x_2, x_3 $(0 \le x_1 \le a, \quad 0 \le x_2 \le b, \quad -h \le x_3 \le h)$ and assumed to be symmetric with respect to the mid-plane $x_3 = 0$, with the face sheet thickness h_f and the core thickness $2h_c$.

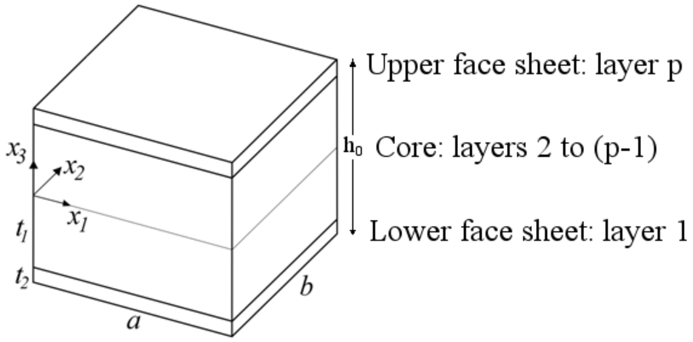

Figure 1. Panel geometry

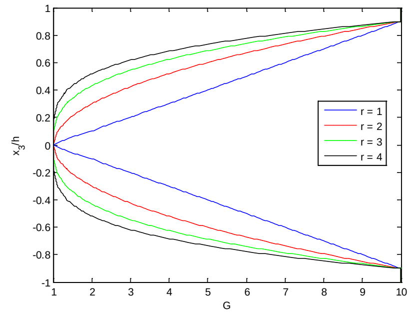

Figure 2. Through-thickness variation of shear modulus for a range of power indices

Let us assume that the panel is divided into p layers $(k = 1,2\ldots, p)$ and that the face sheets (layers 1 & p) and the core, which is subdivided into p-2 layers, are FGMs with constant Poisson's ratios $\nu^{(k)} = const\ (k = 1,\ldots, p)$. The shear moduli of each layer of the core are assumed to vary exponentially through the thickness of the layer from $G^{(k)}$, the value at bottom of the layer to $G^{(k+1)}$ value at top of the layer according to

$$G^{(k)}(x_3) = g^{(k)} \exp\left\{\gamma^{(k)}\left(\frac{x_3}{h} - 1\right)\right\}, \qquad \gamma^{(k)} = \frac{h}{x_3^{(k+1)} - x_3^{(k)}} \ln\left\{\frac{G^{(k+1)}}{G^{(k)}}\right\}, \qquad g^{(k)} = G^{(k)}(h) \quad (1)$$

where $\gamma^{(k)}$ are the inhomogeneity parameters. Through the coefficients of the exponential functions, the number and thickness of the layers, many different variations in through thickness modulus can be modelled such as power law variation [6-8, 10, 12]

$$G(x_3) = G_{tb}\left(\frac{|x_3|}{h_c}\right)^r + G_{core} \quad (2)$$

where $G_{tb} = G_{face} - G_{core}$, G_{face} is the shear modulus of the facesheets and G_{core} is the shear modulus at the centre of the core, and r is the power law index. In order to approximate power law variation in shear moduli using the exponential function, the panel is split into 20 layers of equal thickness (layers 1 and 20 being the facesheets). The shear modulus is calculated at bottom, $G^{(k)}$, and top, $G^{(k+1)}$, of each layer using Eq. (2), and Eq. (1) is then employed to calculate $\gamma^{(k)}$ and $g^{(k)}$ for each layer.

The homogenous face sheets ($k = 1, k = p$) are treated as FGMs with the inhomogeneity parameters, $\gamma^{(k)}$, sufficiently close to zero. They are assumed to be perfectly bonded to the core, so that the continuity of stresses and displacements exists at the face sheet/core interfaces, i.e.

$$x_3 = x_3^{(1)}: \quad \sigma_{i3}^{(2)} - \sigma_{i3}^{(1)} = 0, \quad u_i^{(2)} - u_i^{(1)} = 0, \quad i = 1,2,3 \quad (3a)$$

$$x_3 = x_3^{(p-1)}: \quad \sigma_{i3}^{(p)} - \sigma_{i3}^{(p-1)} = 0, \quad u_i^{(p)} - u_i^{(p-1)} = 0, \quad i = x_1, x_2, x_3 \quad (3b)$$

where $\sigma_{ij}^{(k)}$ and $u_i^{(k)}$ $(k=1,\dots,p)$ are the components of the stress tensor and displacement vector, respectively. The panel is subjected to transverse loading Q on the top surface, so that

$$x_3 = h: \quad \sigma_{33}^{(p)} = Q(x_1, x_2), \quad \sigma_{13}^{(p)} = \sigma_{23}^{(p)} = 0\,. \tag{4a}$$

The loading $Q(x_1, x_2)$ is assumed to allow expansion into a Fourier series

$$Q(x_1, x_2) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{4b}$$

where q_{mn} is the intensity of the loading at the centre of the panel and m and n are wave numbers. The bottom surface of the panel is assumed to be load-free, i.e.

$$x_3 = 0: \quad \sigma_{33}^{(1)} = \sigma_{13}^{(1)} = \sigma_{23}^{(1)} = 0 \tag{5}$$

The Navier-type boundary conditions are assumed at the edges, so that

$$x_1 = 0, a: \quad \sigma_{11}^{(k)} = 0, \quad u_2^{(k)} = u_3^{(k)} = 0, \quad k = 1,\dots,p \tag{6a}$$

$$x_2 = 0, b: \quad \sigma_{22}^{(k)} = 0, \quad u_1^{(k)} = u_3^{(k)} = 0, \quad k = 1,\dots,p \tag{6b}$$

The boundary conditions, Eqs. (6), are representative of roller supports and analogous to simply supported edges in the plate theories (see [13])

Using the displacement function method employed by Kashtalyan and Menshykova [4], the following representation of displacement and stress fields in a sandwich panel with functionally graded core have been obtained:

$$u_1^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} U_{1,jmn}^{(k)}(x_3) \cos\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7a}$$

$$u_2^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} U_{2,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \cos\frac{\pi n x_2}{b} \tag{7b}$$

$$u_3^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} U_{3,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7c}$$

$$\sigma_{33}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{33,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7a}$$

$$\sigma_{13}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{13,jmn}^{(k)}(x_3) \cos\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7b}$$

$$\sigma_{23}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{23,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \cos\frac{\pi n x_2}{b} \tag{7c}$$

$$\sigma_{11}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{11,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7d}$$

$$\sigma_{22}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{22,jmn}^{(k)}(x_3) \sin\frac{\pi m x_1}{a} \sin\frac{\pi n x_2}{b} \tag{7e}$$

$$\sigma_{12}^{(k)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=1}^{6} A_{j,mn}^{(k)} P_{12,jmn}^{(k)}(x_3) \cos\frac{\pi m x_1}{a} \cos\frac{\pi n x_2}{b} \tag{7f}$$

For any pair of m and n, $A_{j,mn}^{(k)}$ are sets of $6 \times p$ arbitrary constants to be determined from Eqs. (3) – (5), $U_{1,jmn}^{(k)}$, $U_{2,jmn}^{(k)}$, $U_{3,jmn}^{(k)}$, $P_{33,jmn}^{(k)}$, $P_{13,jmn}^{(k)}$, $P_{23,jmn}^{(k)}$, $P_{11,jmn}^{(k)}$, $P_{22,jmn}^{(k)}$ and $P_{12,jmn}^{(k)}$ are functions given in [4]. Boundary conditions at the edges of the panel, Eqs. (6), are satisfied exactly.

4. Results and discussion

Validation of the proposed approach is given through comparison with elasticity solutions available in literature. Table 1 shows normalised transverse deflections and stresses in a functionally graded square plate under one-term sinusoidal loading for a range of power law indices *r*. The plate, comprising aluminium and alumina with $E_m = 70GPa$, $E_c = 380GPa$ and constant Poisson's ratio $\nu = 0.3$, has an aspect ratio $\frac{a}{h} = 10$. In all cases normalisation is carried out in an identical fashion to Reddy [14]:

$$\bar{u}_z = u_z(a/2, b/2, h)\left(10h^3E_c/a^4q_0\right); \quad \bar{\sigma}_{xx} = \sigma_{xx}(a/2, b/2, h/2)\left(h/aq_0\right);$$
$$\bar{\sigma}_{xy} = \sigma_{xy}(0, 0, -h/3)\left(h/aq_0\right); \quad \bar{\sigma}_{xz} = \sigma_{xz}(0, b/2, 0)\left(h/aq_0\right).$$

Results of the present 3D piecewise exponential model are in excellent agreement with Reddy [14], which is seen by many researchers as a benchmark method.

r	**Approach**	$\bar{u}_z$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{xy}$
Ceramic	HSDT [14]	0.294	1.98915	0.23778	0.70557
	Present	0.2917	2.0044	0.2383	0.7014
	% difference	0.7823	-0.7667	-0.2187	0.5910
1	HSDT [14]	0.58895	3.08501	0.23817	0.61111
	Present	0.587	3.1052	0.2383	0.6104
	% difference	0.3311	-0.6545	-0.0546	0.1162
2	HSDT [14]	0.75747	3.60664	0.22568	0.54434
	Present	0.7527	3.6208	0.2256	0.544
	% difference	0.6297	-0.3926	0.0354	0.0625
5	HSDT [14]	0.90951	4.24293	0.21609	0.57368
	Present	0.908	4.2538	0.2169	0.5721
	% difference	0.1660	-0.2562	-0.3748	0.2754
Metal	HSDT [14]	1.59724	1.98915	0.23778	0.70557
	Present	1.5833	2.0044	0.2383	0.7014
	% difference	0.8728	-0.7667	-0.2187	0.5910

Table 1. Comparison with Reddy's higher order shear deformation theory [14] for a range of power law indices

Following successful validation, the effect of varying power index *r* on stresses and displacements was explored. The thickness of the face sheets is $h_f = 0.05h_0$ and the Poisson's ratios are taken as $\nu^{(k)} = 0.3, k = 1,...., 20$. Figures 3-6 show through thickness variation of the normalised stresses $\overline{\sigma}_{ij} = \sigma_{ij} / q_{pr}$ and normalised displacements $\overline{U}_i = \dfrac{G_f U_i}{q_{pr} h}$, for four different through thickness variations in stiffness corresponding to $r = 1, 2, 3, 4$. The effect of this variation in stiffness gradient was compared for two sandwich panels both with functionally graded core: the first being a thin panel $(a/h_0 = b/h_0 = 9)$ and the second, a thick panel $(a/h_0 = b/h_0 = 3)$. Through-thickness variation of the normalised transverse shear stress $\overline{\sigma}_{13}$ (Figure 3) shows that increasing the value of power law index *r* (and therefore decreasing the stiffness near the centre of the panel) has the effect of reducing the transverse shear stress in the core. Through-thickness variation of the normalised in-plane normal stress $\overline{\sigma}_{11}$ (Figure 4) is clearly affected by varying power law index *r*. Both in thin and thick panels it can be seen that by increasing power law index *r*, the stresses in the face sheets increase, whilst decreasing in the core. A similar observation can be made for normalised in-plane shear stress $\overline{\sigma}_{12}$.

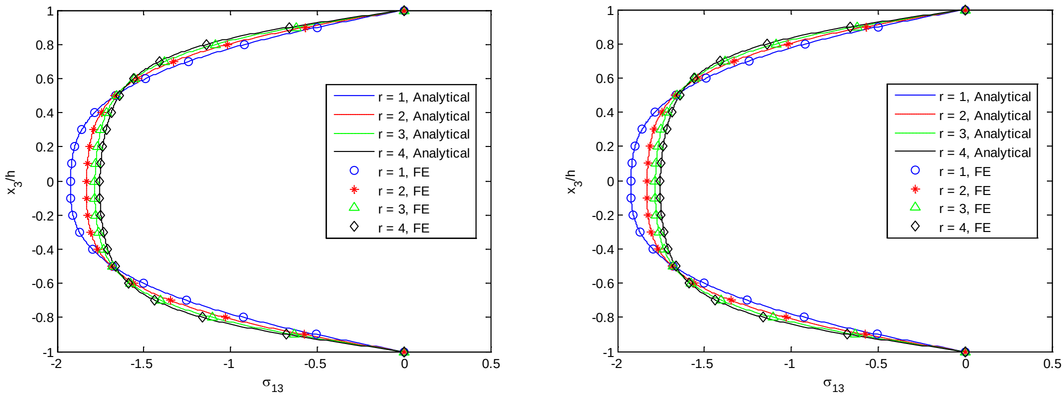

Figure 3. Through-thickness variation of the normalised transverse shear stress $\overline{\sigma}_{13}$ $(0, 0.5b, x_3)$ for a range of power law indexes ($r = 1,2,3,4$) in a thin panel (left) and a thick panel (right)

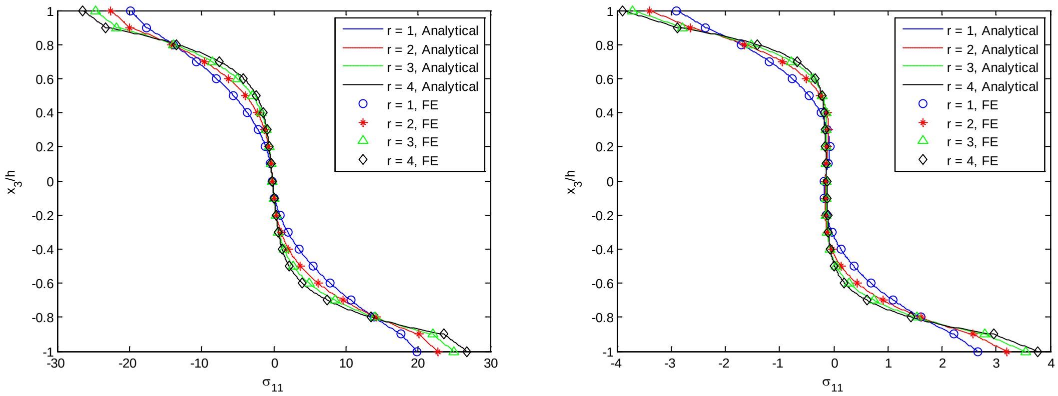

Figure 4. Through-thickness variation of the normalised in-plane normal stress $\overline{\sigma}_{11}$ $(0.5a, 0.5b, x_3)$ for a range of power law indexes ($r = 1,2,3,4$) in a thin panel (left) and a thick panel (right)

Figures 5 and 6 show through-thickness variation of the normalised in-plane displacement $\overline{U}_1$ and normalised transverse displacement $\overline{U}_3$. It can be seen that as the power law index *r* is increased (and therefore the stiffness near the centre of the panel is decreased), that both of these displacement components increase throughout the panel. It should be noted that in-plane displacements are highly non-linear, particularly in thicker panels.

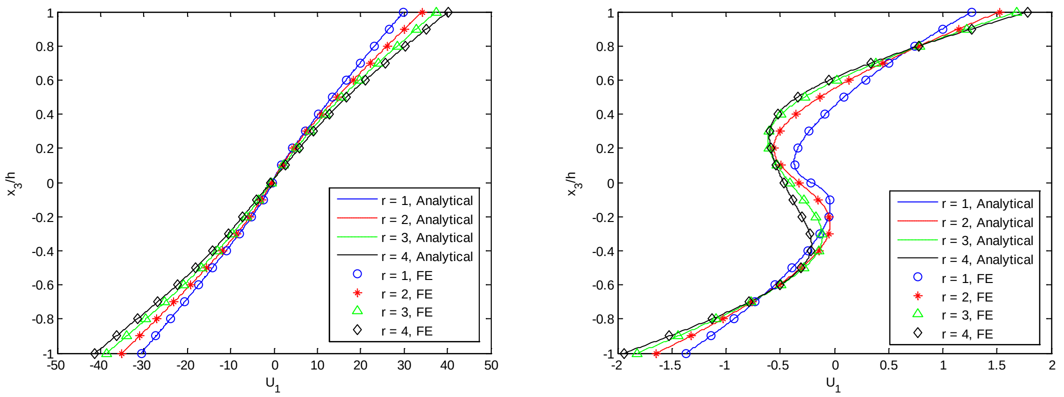

Figure 5. Through-thickness variation of the normalised in-plane displacement $\overline{U}_1\ (0, 0.5b, x_3)$ for a range of power law indexes ($r = 1,2,3,4$) in a thin panel (left) and a thick panel (right)

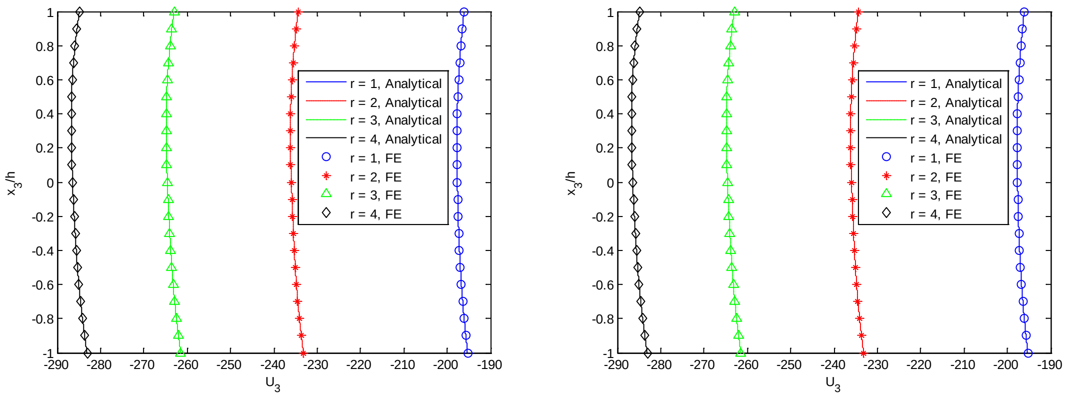

Figure 6. Through-thickness variation of the normalised transverse displacement $\overline{U}_3\ (0, 0.5b, x_3)$ for a range of power law indexes ($r = 1,2,3,4$) in a thin panel (left) and a thick panel (right)

6. Conclusions

The paper presents an investigation into the behaviour of sandwich panels with functionally graded core in the framework of three-dimensional elasticity theory. The current approach makes use of the general solution of the equilibrium equations for inhomogeneous isotropic media obtained earlier [15] and their application to sandwich panels with functionally graded core with exponential variation of the Young's modulus through the thickness [4]. This solution is expanded to produce a piecewise exponential model which can be used to model any functionally graded plate or panel with graded core whose through thickness variation in stiffness can be represented by a smooth function. This new method is fully validated through comparison with results from the literature and a finite element study. As an example, the method is then applied to a sandwich panel whose core exhibits power law variation in stiffness properties and a study is carried out to examine the effect of varying the power law index.

References

[1] Abrate S. *Impact on composite structures* . New York, NY, USA: Cambridge University Press; 1998.

[2] Noda N. Thermal stresses in functionally graded materials. Journal of Thermal Stresses 1999;22(4):477-512.

[3] Anderson TA. A 3-D elasticity solution for a sandwich composite with functionally graded core subjected to transverse loading by a rigid sphere. Composite Structures 2003 6;60(3):265-74.

[4] Kashtalyan M, Menshykova M. Three-dimensional elasticity solution for sandwich panels with a functionally graded core. Composite Structures 2009 1; 87(1):36-43.

[5] Apetre NA, Sankar BV, Ambur DR. Analytical Modeling of Sandwich Beams with Functionally Graded Core. Journal of Sandwich Structures and Materials 2008 January 1;10(1):53-74.

[6] Zenkour AM. Generalized shear deformation theory for bending analysis of functionally graded plates. Appl.Math.Model. 2006;30(1):67-84.

[7] Navazi HM, Haddadpour H, Rasekh M. An analytical solution for nonlinear cylindrical bending of functionally graded plates. Thin-Walled Structures 2006; 44(11):1129-37.

[8] GhannadPour SAM, Alinia MM. Large deflection behavior of functionally graded plates under pressure loads. Composite Structures 2006; 75(1-4):67-71.

[9] Navazi HM, Haddadpour H. Nonlinear cylindrical bending analysis of shear deformable functionally graded plates under different loadings using analytical methods. Int.J.Mech.Sci. 2008;50(12):1650-7.

[10] Yaghoobi H, Fereidoon A. Influence of Neutral Surface Position on Deflection of Funtionally Graded Beam under Uniformly Distributed Load. WASJ 2010; 10(3):337-41.

[11] Tornabene F. Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power-law distribution. Comput.Methods Appl.Mech.Eng. 2009;198(37-40):2911-35.

[12] Alshorbagy AE, Eltaher MA, Mahmoud FF. Free vibration characteristics of a functionally graded beam by finite element method. Appl.Math.Model. 2011; 35(1):412-25.

[13] Kashtalyan M. Three-dimensional elasticity solution for bending of functionally graded rectangular plates. European Journal of Mechanics - A/Solids 2004 10; 23(5):853-64.

[14] Reddy JN. Analysis of functionally graded plates. Int J Numer Methods Eng 2000; 47(1-3):663-84.

[15] Plevako VP. On the theory of elasticity of inhomogeneous media. PMM, vol. 35, No. 5, 1971, pp. 853-860. J.Appl.Math.Mech. 1971; 35(5):806-13.