BRIDGED DEFECTS IN UNI-DIRECTIONALLY-REINFORCED COMPOSITES

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Keywords: crack bridging, unidirectional fibre-reinforced composites, penny-shaped bridged crack, stress-intensity factors

Abstract

Crack-bridging can be encountered in unidirectional fibre-reinforced composites where a matrix crack is constrained by the continuity of fibres across the faces of the crack. The elasticity of the exposed fibres bridging the crack can impose a displacement-dependent traction boundary condition across the faces of the crack. This paper discusses the mathematical modelling of a penny-shaped crack where fibre-elasticity and fibre debonding exert a displacement-dependent normal traction boundary condition across the faces of the crack. The Fredholm integral equation resulting from the formulation of the mixed boundary value problem is solved numerically to evaluate the Mode I stress intensity factor at the boundary of the penny-shaped crack. The influence of fibre elasticity and debonding on K_I *are examined.*

1. Introduction

Unidirectional reinforcement in composites is regarded as a very specific reinforcement configuration adopted to satisfy a functional requirement. It is an idealization since unidirectional reinforcement on its own is rarely used as an engineering solution to enhance the load carrying capacity. Applications generally involve multi-directional reinforcement to address aspects of directional variability of the loading of fibre-reinforced materials. Despite this limitation, the study of unidirectionally reinforced materials provides important insight into micro-mechanical features that can influence the load transfer mechanisms at the scale of fibres. Ideally, in its fabricated condition, fibre-reinforced materials are expected to be defect-free. This is largely a matter of definition, since even perfect fibre-reinforcement can result in micromechanical defects introduced during curing processes and under certain conditions of use, involving localized loads, extreme temperatures and impact loading. The integrity of fibre-reinforced materials can therefore be compromised by the development of features such as fibre breakage, fibre pullout, matrix fracture, fibre-matrix interface delamination, matrix void growth, etc. The importance of damage to the structural integrity of fibre-reinforced materials was discussed several decades ago by a number of researchers and accounts of developments in the area of micromechanical damage are given by Backlund [1] and Selvadurai [2-4]. The topic of flaw- or crack-bridging in unidirectional fibre-reinforced composites was discussed by Kelly [5], Aveston and Kelly [6, 7], Bowling and Groves [8], Sih [9] and Beaumont [10]. The initial investigations dealing with the modelling of flaw-bridging in composites were presented by Selvadurai [2, 3], followed by the work of Stang [11] and several others including Rose, McCartney, Budiansky, Amazigo, Movchan and Willis and others have investigated various aspects of the elastostatic problem of bridging-induced behaviour of flaws in unidirectional fibre-reinforced materials. References to these works can be found in Selvadurai [12]

This paper examines the flaw-bridging problem in a unidirectionally reinforced elastic composite, where the bridging occurs as a result of the development of a crack in the matrix and the intact fibres that are intact and continuous across the faces of the crack, which exert a displacement-dependent traction constraint on the crack surfaces. The types of cracks that can develop in a unidirectionally reinforced can be varied and these can include cracks that are present at the boundary of a unidirectionally reinforced composites to through cracks that can occur across unidirectionally reinforced plates to bridged cracks that can be present at the interior of uni-directionally reinforced composites. The orientation of such cracks can also be varied and cannot be determined with certainty. The effectiveness of the bridging action is expected to be reduced if the fibre-bridging occurs at an inclination to the plane of such matrix cracks. It is convenient to examine the role of flaw bridging by considering the problem of the penny-shaped matrix crack that is located normal to the direction of uni-directional reinforcement. The presence of fibre continuity changes the character of the integral equation governing the stress analysis problem from an Abel to a Fredholm type. This integral equation can be solved numerically to determine the influence of the flaw-bridging on the Mode I stress intensity factor. The paper reviews the formulation of the flaw-bridging problem related to a e penny-shaped crack and an external circular crack and examines the limiting cases of fibre-bridging on the Mode I stress intensity factor.

2. Basic results for transverse isotropic elastic materials

We consider the class of axisymmetric problems related to an elastic composite, which is reinforced uni-directionally by elastic fibres. We assume that the effective transversely isotropic elastic properties of the fibre-reinforced material can estimated by recourse to theories of elastic composites similar to those as proposed by Hashin and Rosen [13], Hill [14], Broutman and Krock [15], Hale [16], Christensen [17] and Spencer [18] (see also Selvadurai and Nikopour [19-21]). We consider the mechanics of the transversely isotropic elastic materials where the axis of elastic symmetry coincides with the fibre direction. As shown by Elliott [22] and Shield [23], the displacement and stress fields in the resulting transverse isotropic elastic material can be expressed in terms of two functions $\varphi_i(r,z)$, $(i=1,2)$, which are solutions of

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z_i^2}\right)\varphi_i(r,z)=0 \qquad (1)$$

where $z_i = z/\sqrt{\nu_i}$, and ν_i are roots of the equation

$$c_{11}c_{44}\nu^2+[c_{13}(2c_{44}+c_{13})-c_{11}c_{33}]\nu+c_{33}c_{44}=0 \qquad (2)$$

and c_{ij} are the elastic constants of the transversely isotropic elastic material. In general

$$c_{ij} = c_{ij}(E_f, E_m, \nu_f, \nu_m, V_f, V_m, C) \tag{3}$$

where the material parameters of the composite depend on the constituent properties , the volume fractions and a contiguity factor, C. Explicit expressions for c_{ij} are given, for example, in [13], [17] and [19]. The displacement and stress fields relevant to the problems considered here are

$$\{u_z(r,z)\,;\sigma_{zz}(r,z)\,;\sigma_{rz}(r,z)\,\} = \sum_{i=1}^{2} \left\{ \frac{\partial}{\partial z} k_i \varphi_i\,; \quad (k_i c_{33} - \nu_i c_{13}) \frac{\partial^2 \varphi_i}{\partial z^2}\,; \quad c_{44}(1+k_i) \frac{\partial^2 \varphi_i}{\partial r \partial z} \right\} \tag{4}$$

where

$$k_i = (c_{11}\nu_i - c_{44})/(c_{13} + c_{44}) \tag{5}$$

3. The Penny-Shaped Bridged Crack

We examine the problem of a penny-shaped matrix crack that is located in a uni-directional fibre-reinforced material, where the fibres exhibit continuity across the faces of the crack. The plane of the penny-shaped matrix crack is located normal to the direction of uniaxial reinforcement, and it is assumed that the bulk elastic behaviour of the composite can be modelled as a transversely isotropic elastic material. The penny-shaped matrix crack of radius b interacts with the external loading by imposing additional displacement-dependent tractions on the face of the matrix crack. The tractions that develop can be influenced by a variety of factors including fibre debonding at the crack face, fibre yielding, frictional slip, etc. The analysis of penny-shaped cracks with displacement-dependent traction boundary conditions on the crack faces was first examined by Atkinson [24] using an iterative technique. Even though fibre continuity is present, the reduction of the bridging fibre length to zero effectively introduces infinite bridging stiffness to the crack tip, which suppresses the development of the singular stress field. The approach adopted by Selvadurai [2-4] assumes that debonding occurs uniformly over the faces of the crack such that the exposed length of the fibres is $2l$ (see inset sketch in Figure 1). This enables the mathematical formulation of the bridged penny-shaped crack problem with a displacement-dependent traction constraint. We consider the problem where the composite containing the bridged crack is subjected to an external stress state which is symmetric about the z-axis, and aligned with the fibre direction. The axisymmetric bridged crack problem can be formulated in relation to a halfspace region, where the surface of the halfspace region is subjected to mixed boundary conditions

$$u_z(r,0) = 0\,, \quad b \le r < \infty \qquad ; \quad \sigma_{rz}(r,0) = 0 \quad ; \quad 0 \le r \le \infty \tag{6}$$

$$\sigma_{zz}(r,0) = -p^*(r) + \frac{E_f V_f}{l} u_z(r,0) \quad ; \quad 0 < r < b \tag{7}$$

where $p^*(r)$ is the tensile traction induced in the plane $z = 0$ of the intact composite due to the action of the external stress state. For the analysis of the mixed boundary value problem posed by (6) and (7), we seek solutions of (1), which are based on Hankel transform developments (Sneddon [25]). The relevant solutions that satisfy the regularity conditions applicable to a halfspace region are

$$\varphi_i(r,z) = \frac{1}{b^2}\int_0^\infty \xi\, A_i(\xi)\, e^{-\eta_i z}\, J_0(\xi r/b)\, d\xi \tag{8}$$

where $A_i(\xi)$ are arbitrary functions and $\eta_i = \xi / b\sqrt{\nu_i}$. The mixed boundary conditions (6) and (7) can be reduced to system of dual integral equations for a single unknown function. Using a finite Fourier transform, we can further reduce the dual system to a single Fredholm integral equation of the second-kind for an unknown function $\phi(t)$, which takes the form:

$$\phi(t) - \frac{\beta}{\pi}\int_0^1 K(t,\tau)\phi(\tau)\, d\tau = g(t) \tag{9}$$

where

$$\begin{aligned} K(t,\tau) &= 2\int_0^\infty \vartheta^{-1}\sin(\vartheta t)\sin(\vartheta\tau)\, d\vartheta \\ \beta &= \frac{E_f V_f b\sqrt{\nu_1\nu_2}\,(k_1-k_2)}{E_m l\,\Omega^*}\,; \quad \Omega^* = \frac{\Omega\, c_{44}}{E_m} \\ \Omega &= \sqrt{\nu_1}(1+k_1)\left(\frac{k_2c_{33}-\nu_2c_{13}}{c_{44}}\right) - \sqrt{\nu_2}(1+k_2)\left(\frac{k_1c_{33}-\nu_1c_{13}}{c_{44}}\right) \end{aligned} \tag{10}$$

The function $g(t)$ depends only on the nature of the axisymmetric external loading. For example, when the composite is subjected to a uniform tensile stress field at infinity

$$g(t) = t \tag{11}$$

It should be noted the function $\phi(t)$ will contain a multiplier that will take into account the magnitude and the nature of the loading. The mathematical analysis of the bridged penny shaped crack problem (for $t \in \text{real}$ and $\tau \in \text{real}$) is formally reduced to the solution of

$$\phi(t) - \frac{\beta}{\pi}\int_0^1 \log_e\left(\frac{t+\tau}{t-\tau}\right)\phi(\tau)\, d\tau = g(t) \quad ; \quad t > \tau \geq 0 \tag{12}$$

The integral equation (12) can be classified as a Fredholm type, even though the kernel suffers a discontinuity at $t = \tau$, since the kernel is quadratic integrable (Mikhlin [26]): i.e.

$$\int_0^1\int_0^1 \log_e^2|t-\tau|\, dt\, d\tau \to \text{finite}$$

The solution of (12) provides, formally, results of importance to the idealized bridged penny-shaped crack with a bridged region of constant length $2l$ over the entire crack surface. The result of particular interest to fracture mechanics of composites relates to the Mode I stress intensity factor at the crack tip, defined by

$$\mathrm{K_I} = \lim_{r\to b^+}[2(r-b)]^{1/2}\sigma_{zz}(r,0) \tag{13}$$

Considering the result for the axial stress expressed in terms of $\phi(t)$, it can be shown that for a penny-shaped crack in a uni-directional fibre reinforced material with crack-bridging and subjected to a uniform far-field axial stress σ_0,

$$\mathrm{K_I} = [2\sigma_0 \sqrt{b} / \pi]\phi(1) \tag{14}$$

In the limiting case when the elasticity of the bridging fibres $E_f \to 0$, $\beta \to 0$ and we have a penny-shaped crack located in a matrix with uni-directional cavities where originally there were fibres. Since the resulting material is still transversely isotropic and $\phi(1) = 1$, the expression (14) for the stress intensity factor gives

$$\mathrm{K_I} = 2\sigma_0 \sqrt{b} / \pi \tag{15}$$

which is the classical result. The stress intensity factor is independent of the transverse isotropy of the medium with directional voids. Consider the limiting case when the uni-directional fibre-reinforced material is reinforced with inextensible fibres (i.e. $E_f \to \infty$): This is an idealization that was proposed by Adkins and Rivlin [27] and successfully developed by Spencer [28, 29] and co-workers for examining a wide class of problems in fibre-reinforced materials. In the limit, the integral equation (12) reduces to

$$\int_0^1 \log_e\left(\frac{t+\tau}{t-\tau}\right)\phi(\tau)\, d\tau = 0 \tag{16}$$

which has a trivial solution $\phi(t) = 0$. Consequently, $\mathrm{K_I} \equiv 0$, and the stress intensity factor is completely suppressed. The limit, of course, has to be approached with caution since there are boundary layers that can exist in the medium at the crack tip, which can lead to stress channeling phenomena. For arbitrary values of the elastic properties of the fibre-reinforced composite, the integral equation (12) has a non-degenerate solution. There appears to be no closed form solution of this equation and the Fredholm integral equation can be solved using quadrature techniques that reduce the integral equation to a matrix equation. Details of the method are well documented in the literature (Baker [30], Delves and Mohamed [31], Atkinson [32], Selvadurai [33, 34]). Figure 1 illustrates the influence of the fibre-matrix elastic modular ratio and the geometry of the bridging zone on the stress intensity factor for the penny-shaped crack.

4. Concluding remarks

The results presented in the paper demonstrate that the presence of fibre-continuity across the faces of a defect can influence the development of stress intensity factors at the crack tip. This influence arises from two effects; the first is as a result of the elasticity of the bridging fibres and the second is due to the debonded length of the fibres in the vicinity of the crack. The accurate determination of the latter parameter is a difficult exercise. The paper considers a debonded region of uniform length that facilitates the mathematical modelling of the problem and enables the evaluation of the stress intensity factors at the crack tip. In the limiting case when the fibres have relatively low elastic modulus compared to the matrix the Mode I stress intensity factor at the crack tip reduces to that for the classical problem of a penny-shaped crack in a transversely isotropic elastic solid and when the fibres have a higher relative stiffness the Mode I stress intensity factor is suppressed. These conclusions are derived from the numerical evaluation of the governing integral equation.

Acknowledgements

The work described in this paper was supported by a *Discovery Grant* awarded by the *Natural Sciences and Engineering Research Council of Canada* and through the *James McGill Research Chairs Program* at McGill University.

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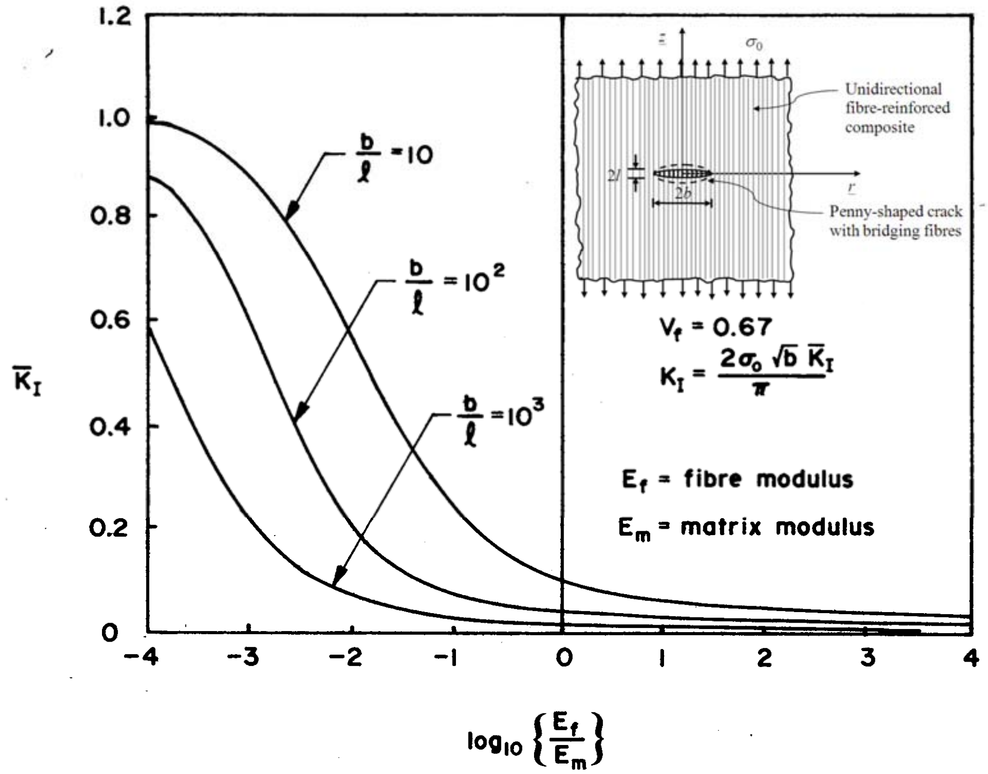

Figure 1. Normalized Mode I stress intensity factor for a bridged penny-shaped crack [$\bar{\mathrm{K}}_\mathrm{I} = \mathrm{K}_\mathrm{I} /(2\sigma_0\sqrt{b}/\pi)$