ANALYTICAL HOMOGENIZATION OF HONEYCOMB WITH SKIN AND HEIGHT EFFECTS IN SANDWICH PLATES

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Abstract
The honeycomb sandwich plates are widely used in the automotive, aeronautic and aerospace industries. However, the numerical modeling of honeycomb structures is too tedious and time consuming. The homogenization of these structures allows obtaining an equivalent homogenized solid and making very efficient simulations. In the present study, the skin effect is taken into consideration for the extension problems of honeycomb sandwich plates in which the two skins are much more rigid than the honeycomb core. An analytic homogenization method based on the beam and plate theories is proposed to determine the upper and lower bounds of the equivalent elastic properties, and to study the influence of the honeycomb height on these properties. A very good agreement has been achieved between the results of the present H-model and 3D FE modeling.

1. Introduction

The honeycomb sandwich plates are widely used in the automotive, aeronautic and aerospace industries. The simulation and optimization of this kind of plates are of prime importance for the lightness and safety of composite structures. However, the numerical modeling of honeycomb structures is too tedious and time consuming. The homogenization of these structures allows obtaining an equivalent homogeneous solid and making the simulations much more efficient.

Many studies have been performed on the analytical homogenization of honeycomb structures. The book of Gibson and Ashby [1] is the first systematic literature in this field. The in-plane elastic properties of honeycomb were first obtained with the beam theory. Further refinements have been attempted by Masters and Evans [2] considering stretching and hinging effects. However, all these mathematical models on honeycomb cores are based on pure cellular structures without considering the strengthening effect of the skin faces. In the classical sandwich theory [3], the global skin-core interaction is identified as the result of the anti-plane core assumption. Since the constraints of two skin faces significantly alter the local deformation mechanism of the core, the homogenized core properties become sensitive to the ratio of the core thickness over the unit cell size, which is called thickness effect by Becker [4].

An interesting approach was proposed by Xu *et al.* [5] to homogenize a honeycomb unit cell including skin effect. Firstly, the homogenization was carried out along *X*-direction (Fig. 1) to obtain 2 types of solids corresponding to inclined and vertical walls; then a second homogenization was carried out along *Y*-direction to obtain the whole homogenous solid. Based on the asymptotic expansions and characteristic periodicity, the displacement functions between the 2 types of solids were formulated and analytically resolved. However, the interactions between the two homogenized solids were not equivalent to those between the vertical and inclined walls; furthermore the Poisson coefficients were not treated together with the tensile moduli, leading to some notable errors.

In the present study, we limit ourselves to the extension problems of honeycomb sandwich plates in which the two skins are much more rigid than the honeycomb core. Thus, we can assume that: 1) the core has little influence on the skin's behavior, and its deformation is constrained by the two skins; 2) for a honeycomb in thin walls, the tensile rigidities are essentially given by the core's paper stretching instead of its bending. In a representative unit cell, two analytical homogenization models based on the beam and plate theories are established to determine the upper and lower bounds of the elastic properties of the equivalent solid. An analytical homogenization model based on the plate theory and trigonometric function series is proposed to study the influence of the honeycomb height on its elastic properties. The coupling of the tensile moduli and Poisson coefficients is considered to study the behavior interactions in two directions. A very good agreement has been achieved between our H-model and 3D FE modeling results.

2. Formulation of the homogenization for in-plane tensile properties

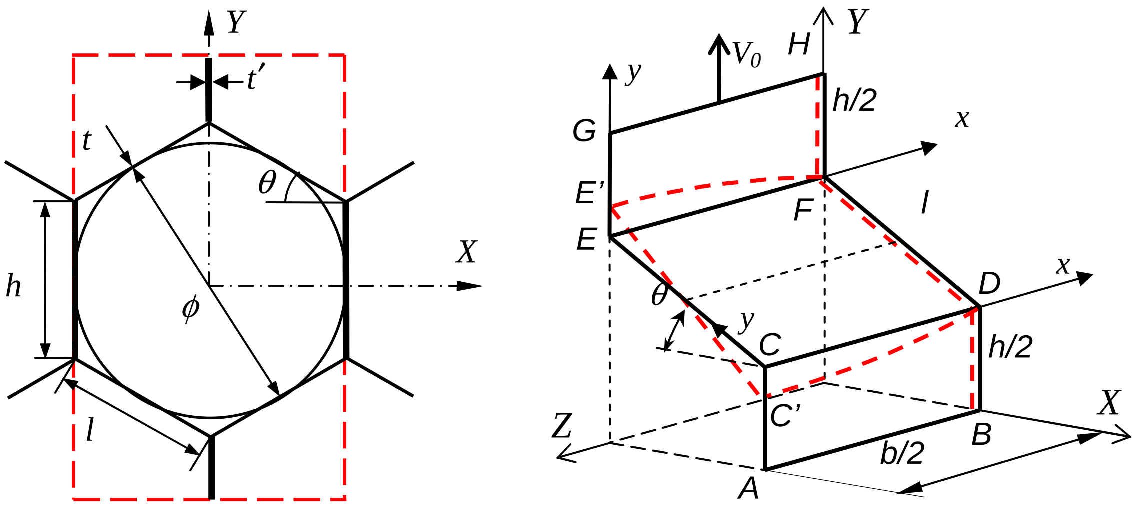

Figure 1. Honeycomb REV and its stretching behavior on 1/8 REV

A honeycomb cell (Fig. 1) is taken as Representative Elementary Volume (REV). In the classical homogenization theory [1], the tensile properties are determined only on a cell without the skin effect, and the properties depend only on the bending behavior of the thin walls of the honeycomb. In the present study, the skins are supposed very rigid, so the honeycomb walls are constrained by the skins, the thin wall stretching effect is dominant with respect to its bending effect. Consequently, the tensile moduli of the honeycomb cell are rather proportional to t/l (Fig. 1), instead of $(t/l)^3$. Taking $t = 0.19$ mm and $l = 8$ mm as example, the tensile moduli with the skin effect are 591 times of those without skin effect! The stretching behavior of the honeycomb under a tensile load in *Y*-direction is shown in Fig. 1.

2.1 Determination of the upper bounds of the tensile properties

When the honeycomb height is very small, one can suppose that all material points on the honeycomb walls behave as the points on the skins, so have the same constant strains (Fig. 1):

$$\varepsilon_Y \text{ (given strain)} \quad ; \quad \varepsilon_X = -\nu_S \varepsilon_Y \quad ; \quad \varepsilon_l = \left(\sin^2\theta - \nu_S \cos^2\theta\right)\varepsilon_Y = \alpha\varepsilon_Y \qquad (1)$$

where ν_S is the skin's Poisson coefficient, ε_l is the normal strain in the inclined walls.

The above assumption allows one to determine the upper bounds of the tensile modulus. Considering the membrane behavior of the honeycomb walls in Fig. 1, the internal strain energy is expressed as follows by using local references:

$$\pi_{\text{int}} = \frac{1}{2}\frac{E}{1-\nu^2} bth\left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y\right) + \frac{1}{2}\frac{E}{1-\nu^2} btl\left(\varepsilon_x^2 + \varepsilon_l^2 + 2\nu\varepsilon_x\varepsilon_l\right) \quad \text{with} \quad \varepsilon_y = \varepsilon_Y \qquad (2)$$

where E is the Young's modulus of the honeycomb paper, ν is its Poisson coefficient, ε_x is the normal strain due to the Poisson effect and has the same value on the vertical and inclined walls due to the rigid skin effect. ε_x can be obtained by minimizing the strain energy, then the strain energy of the honeycomb (Eq. 2) can be calculated for a given ε_Y :

$$\frac{\partial \pi_{\text{int}}}{\partial \varepsilon_x} = 0 \quad \Rightarrow \quad \varepsilon_x = -\nu\frac{h+l\alpha}{h+l}\varepsilon_Y \quad \Rightarrow \quad \pi_{\text{int}} \qquad (3)$$

The strain energy for an extension along *X*-direction can be obtained in a similar manner. The tensile strain imposed in the *X*-direction involves the following strains in the vertical and inclined walls:

$$\varepsilon_X \text{ (given strain)} \quad ; \quad \varepsilon_Y = -\nu_S \varepsilon_X \quad ; \quad \varepsilon_l = \left(\sin^2\theta - \cos^2\theta / \nu_S\right)\varepsilon_Y = \beta\varepsilon_Y \qquad (4)$$

By using Eq. (4) and the same method, the strain energy of the honeycomb can be calculated for a given ε_X .

The equivalent homogenized solid of the honeycomb (indicated by *) can be considered as an orthotropic material, having the following elastic law:

$$\begin{Bmatrix} \sigma_X^* \\ \sigma_Y^* \end{Bmatrix} = \frac{1}{1-\nu_{XY}^*\nu_{YX}^*}\begin{bmatrix} E_X^* & \nu_{XY}^* E_Y^* \\ \nu_{YX}^* E_X^* & E_Y^* \end{bmatrix}\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \end{Bmatrix} = \begin{bmatrix} Q_X^* & Q_{XY}^* \\ Q_{XY}^* & Q_Y^* \end{bmatrix}\begin{Bmatrix} \varepsilon_X \\ \varepsilon_Y \end{Bmatrix} \quad \text{with} \quad \frac{E_X^*}{\nu_{XY}^*} = \frac{E_Y^*}{\nu_{YX}^*} \qquad (5)$$

The above elastic matrix is symmetrical one, so only three parameters Q_X^*, Q_Y^*, Q_{XY}^* have to be determined. It is noted that in most honeycomb homogenization studies, the tensile moduli and Poisson coefficients were determined separately without respecting Eq. (5).

The powerful energy method is used to determine these properties. The strain energy of the homogenized solid is defined as follows and should be equal to the strain energy of the real honeycomb:

$$\begin{aligned}\pi^*_{\text{int}} &= \frac{1}{2}\int_V \left(\varepsilon_X^2 Q_X^* + 2\varepsilon_X \varepsilon_Y Q_{XY}^* + \varepsilon_Y^* Q_Y^*\right) dXdYdZ \\ &= \frac{1}{2}\left(\varepsilon_X^2 Q_X^* + 2\varepsilon_X \varepsilon_Y Q_{XY}^* + \varepsilon_Y^* Q_Y^*\right) V = \pi_{\text{int}} \qquad \text{with} \quad V = \frac{1}{2} bl\cos\theta(h + l\sin\theta)\end{aligned} \tag{6}$$

so $$\varepsilon_X^2 Q_X^* + 2\varepsilon_X \varepsilon_Y Q_{XY}^* + \varepsilon_Y^* Q_Y^* = \frac{2}{V}\pi_{\text{int}} \tag{7}$$

The three unknown parameters Q_X^*, Q_Y^*, Q_{XY}^* need three equations like (7), which can be obtained by three numerical tests. For example, an extension along *X* (ν_S=0.3, ε_X, ε_Y and ε_l defined by Eqs 1-2), another along *Y* (ν_S=0.3, ε_X, ε_Y and ε_l defined by Eq. 4) and the third along *Y* with ν=0. It is noted that the results do not depend on the choice of the three cases.

Using the obtained Q_X^*, Q_Y^*, Q_{XY}^*, one can calculate the Poisson coefficients and Young's moduli as follows:

$$\nu_{YX}^* = \frac{Q_{XY}^*}{Q_X^*} \quad ; \quad \nu_{XY}^* = \frac{Q_{XY}^*}{Q_Y^*} \quad ; \quad E_X^* = Q_X^*\left(1 - \nu_{XY}^* \nu_{YX}^*\right) \quad ; \quad E_Y^* = Q_Y^*\left(1 - \nu_{XY}^* \nu_{YX}^*\right) \tag{8}$$

2.2 Determination of the lower bounds of the tensile modulus

When the honeycomb is very high, most material points on its walls behave as the points located at the mid-height (points A, *C*, *E*, *G* in Fig. 1), leading to a redistribution of the strains and stresses between the vertical and inclined walls. This case gives the lower bounds of the tensile moduli. It is noted that the skins only constrain the displacements of the honeycomb walls on the REV borders (Fig. 1, left). Grouping the two half vertical walls together for simplicity (Fig. 2), an imposed displacement V_0 on the top of the 1/4 REV leads to the following deformations:

$$\varepsilon_Y = \frac{V_0}{h + l\sin\theta} \quad ; \quad \varepsilon_X = -\nu_S \varepsilon_Y \quad ; \quad U = \varepsilon_X l\cos\theta = -\nu_S \frac{V_0 l\cos\theta}{h + l\sin\theta} \tag{9}$$

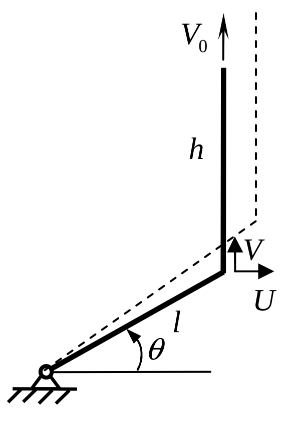

Figure 2. Strain and stress redistribution between the vertical and inclined walls in 1/4 REV

It is supposed that the stress redistribution gives a constant vertical displacement V along the honeycomb height direction. Thus the strains on the vertical and inclined walls are given by:

$$\varepsilon_h = \frac{1}{h}\left(V_0 - V\right) \quad ; \quad \varepsilon_l = \frac{1}{l}\left(U\cos\theta + V\sin\theta\right) \tag{10}$$

In the case of a very high honeycomb, the skin constraint effect on the Poisson behavior in the height direction is negligible, thus the strain energy is defined as follows:

$$\pi_{\text{int}} = \frac{1}{2}Ebt\left(\varepsilon_h^2 h + \varepsilon_l^2 l\right) \tag{11}$$

The unknown displacement V in Eq. (10) is obtained by minimizing the strain energy; then the strain energy can be calculated:

$$\frac{\partial \pi_{\text{int}}}{\partial V} = 0 \quad ; \quad V = \frac{lV_0 - hU\sin\theta\cos\theta}{h\sin^2\theta + l} \quad \rightarrow \quad \pi_{\text{int}} \tag{12}$$

For an extension along X-direction, a displacement U_0 is imposed, leading to the following deformations:

$$\varepsilon_X = \frac{U}{l\cos\theta} \quad ; \quad \varepsilon_Y = -\nu_S \varepsilon_X \quad ; \quad V_0 = -\nu_S \frac{U\left(h + l\sin\theta\right)}{l\cos\theta} \tag{13}$$

Thus we can use Eq. (13) and above method to calculate the strain energy. Finally, the lower bounds of the tensile moduli and the Poisson coefficients can be obtained by using the energy method described by Eqs (5-8).

2.3 Honeycomb height effect on the tensile moduli and Poisson coefficients

In §2.2, the honeycomb height is respectively supposed very small or very great to determine the upper and lower bounds of the tensile properties. Let us study now the Honeycomb height effect on the tensile moduli and Poisson coefficients, including the in plane shear strain contribution.

1) On the upper vertical wall (h/2)

The honeycomb height has a big influence on the tensile moduli and Poisson coefficients of the honeycomb. The points on the walls have not only the same displacements as the skins, but also some additional displacements which are supposed as follows in the local reference xy to satisfy the boundary conditions (Fig. 1, right):

$$\begin{cases} u_1 = \sum\limits_{i=1}^{n} a_i \sin\dfrac{j\pi x}{b} \\ v_1 = \sum\limits_{i=1}^{n} b_i \cos\dfrac{j\pi x}{b}\left(1 - \dfrac{2y}{h}\right) \end{cases} \quad ; \quad j = 2i - 1 \tag{14}$$

It is worth mentioning that the trigonometric functions in u_1 can be replaced by a linear function ($2a_1x/b$) in the case of tensile loading. Including the imposed constant skin's strain ε_Y, the strains in the local reference xy can be obtained as follows:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u_1}{\partial x} & = \displaystyle\sum_{i=1}^{n} a_i \frac{j\pi}{b} \cos \frac{j\pi x}{b} \\ \varepsilon_y = \varepsilon_Y + \dfrac{\partial v_1}{\partial y} & = \varepsilon_Y - \dfrac{2}{h} \displaystyle\sum_{i=1}^{n} b_i \cos \frac{j\pi x}{b} \\ \gamma_{xy} = \dfrac{\partial u_1}{\partial y} + \dfrac{\partial v_1}{\partial x} & = -\displaystyle\sum_{i=1}^{n} b_i \frac{j\pi}{b} \sin \frac{j\pi x}{b} (1 - \frac{2y}{h}) \end{cases} \tag{15}$$

Thus the strain energy in a vertical wall (h/2) $\pi_{\text{int\_}h}$ is expressed as follows:

$$\pi_{\text{int\_}h} = t\int_0^{b/2}\int_0^{h/2}\left[\frac{Et}{1-\nu^2}\left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y\right) + Gt\gamma_{xy}^2\right]dydx = \frac{Et}{1-\nu^2}\left(A_1 + B_1 + 2\nu C_1\right) + GtD_1 \tag{16}$$

with

$$\begin{aligned} A_1 &= \sum_{i=1}^{n} \frac{j^2\pi^2}{8}\frac{h}{b} a_i^2 \quad ; \quad B_1 = \frac{bh}{4}\varepsilon_Y^2 + 2\varepsilon_Y \sum_{i=1}^{n} b_i \frac{b}{j\pi}(-1)^i + \frac{1}{2}\frac{b}{h}\sum_{i=1}^{n} b_i^2 \\ C_1 &= \sum_{i=1}^{n} -\frac{h}{2} a_i \varepsilon_Y \left(-1\right)^i - \frac{j\pi}{4} a_i b_i \quad ; \quad D_1 = \frac{h}{b}\sum_{i=1}^{n} b_i^2 \frac{j^2\pi^2}{24} \end{aligned} \tag{17}$$

2) On 1/2 inclined wall (l/2)

In the local reference of the inclined wall, the same additional displacements are observed (Fig.1, right, $\overrightarrow{CC'} = -\overrightarrow{EE'}$), which should be projected on the inclined wall plane. Thus the displacements and strains can be defined as follows:

$$\begin{cases} u_2 = \displaystyle\sum_{i=1}^{n} a_i \sin \frac{j\pi x}{b} \\ v_2 = -\displaystyle\sum_{i=1}^{n} b_i \sin\theta \cos \frac{j\pi x}{b} (1 - \frac{2y}{l}) \end{cases} \quad ; \qquad j = 2i - 1 \tag{18}$$

$$\begin{cases} \varepsilon_x = \dfrac{\partial u_2}{\partial x} & = \displaystyle\sum_{i=1}^{n} a_i \frac{j\pi}{b} \cos \frac{j\pi x}{b} \\ \varepsilon_y = \varepsilon_l + \dfrac{\partial v_2}{\partial y} & = \varepsilon_l + \dfrac{2}{l} \displaystyle\sum_{i=1}^{n} b_i \sin\theta \cos \frac{j\pi x}{b} \\ \gamma_{xy} = \dfrac{\partial u_2}{\partial y} + \dfrac{\partial v_2}{\partial x} & = -\displaystyle\sum_{i=1}^{n} b_i \frac{j\pi}{b} \sin\theta \sin \frac{j\pi x}{b} (1 - \frac{2y}{l}) \end{cases} \tag{19}$$

Consequently, the strain energy in the inclined wall $\pi_{\text{int\_}l}$ is given by:

$$\pi_{\text{int\_}l} = \frac{Et}{1-\nu^2}\left(A_2 + B_2 + 2\nu C_2\right) + GtD_2 \tag{20}$$

where
$$A_2 = \sum_{i=1}^{n} \frac{j^2\pi^2}{8}\frac{l}{b} a_i^2 \quad ; \quad B_2 = \frac{bl}{4}\varepsilon_l^2 - 2\varepsilon_l \sin\theta \sum_{i=1}^{n} b_i \frac{b}{j\pi}(-1)^i + \frac{1}{2}\frac{b}{l}\sin^2\theta \sum_{i=1}^{n} b_i^2$$
$$C_2 = \sum_{i=1}^{n} -\frac{h}{2} a_i \varepsilon_l (-1)^i + \frac{j\pi}{4}\sin\theta a_i b_i \quad ; \quad D_2 = \frac{l}{b}\sin^2\theta \sum_{i=1}^{n} b_i^2 \frac{j^2\pi^2}{24} \tag{21}$$

The minimization of the strain energy gives a linear equation system, thus one obtains the unknown parameters a_i and b_i, and then the strain energy:

$$\frac{\partial(\pi_{\text{int\_}h} + \pi_{\text{int\_}l})}{\partial a_i} = 0 \quad ; \quad \frac{\partial(\pi_{\text{int\_}h} + \pi_{\text{int\_}l})}{\partial b_i} = 0 \quad \Rightarrow \quad a_i \text{ and } b_i \quad \Rightarrow \quad \pi_{\text{int}} \tag{17}$$

For an extension along *X*-direction, one should just replace the Eq. (1) by (4). Three strain states are taken to establish three equations like Eq. (7) in order to calculate Q_X^*, Q_Y^*, Q_{XY}^* and ν_{XY}^*, ν_{YX}^*.

3. Numerical validation

The present H-models are validated by FE simulations using the thin shell element 'S4R' of Abaqus. The material and geometrical data are given as follows: for the skins, E_S=10000 MPa, t_S=0.6 mm, ν_S=0.3; for the honeycomb paper, E=1500 MPa, t=0.19 mm, ν=0.3, h=l=4.62 mm, θ=30°.

Since the tensile rigidities of the honeycomb are very small with respect to those of the skins, two simulations are carried out each time by using E_1=1000 MPa and E_2=2500 MPa for the honeycomb paper, then the strain energy is obtained by subtraction in order to accurately calculate the tensile properties.

The tensile moduli Q_X^*, Q_Y^*, Q_{XY}^* vs the honeycomb height are shown in Fig. 3, it is observed that the present analytical curves of the tensile moduli are very close to the numerical ones obtained by Abaqus. These curves are well situated between the upper and lower bounds. It is noted that the honeycomb height has a great influence on the tensile moduli: Q_X^* decreases from 28 to 21.4 MPa and Q_Y^* decreases from 52 to 21.4 MPa. A very good agreement is also found for the Poisson coefficients ν_{XY}^* and ν_{YX}^* between the H-model and Abaqus (Fig. 4).

It is also worth noting that when the honeycomb height increases, the three moduli Q_X^*, Q_Y^*, Q_{XY}^* decrease and converge to the same lower bounds (Fig. 3), and the Poisson coefficients increase and converge toward 1 (Fig. 4). This means that there are strong interactions between both directions: an extension along a direction may induce considerable stresses in another direction, this needs much more energy so leads to a greater tensile modulus. Notable differences are found compared to the results obtained in [5], because the honeycomb was not treated as an entire 3D structure and the moduli E_X^*, E_Y^*, ν_{XY}^*, ν_{YX}^* were calculated separately without considering their coupling effects in [5].

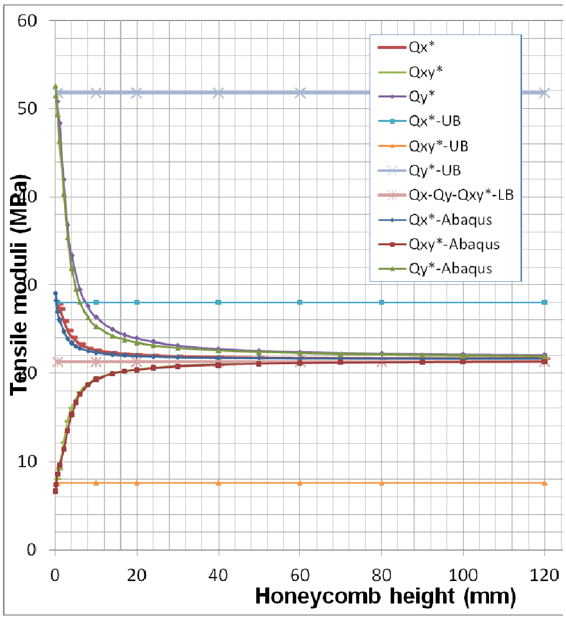

Figure 3. Tensile moduli vs height (b=0-120 mm)

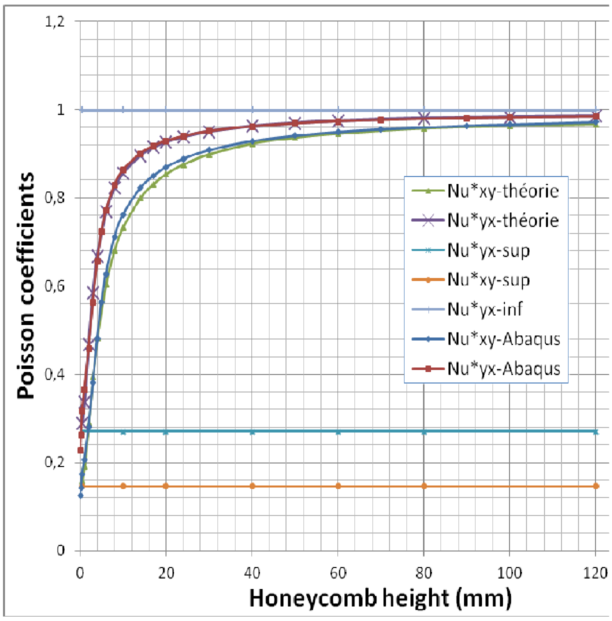

Figure 4. Poisson coefficients vs height (b=0-120 mm)

4. Conclusions

Analytical homogenization models of honeycomb sandwiches plates are developed based on the strain energy method. The skin effect and honeycomb height effect on the tensile modulus are investigated, leading to a great improvement compared to the classical homogenization models. The coupling effect between the Young's moduli and Poisson coefficients is studied. The obtained homogenized tensile moduli and the Poisson coefficients are in very good agreement with those got from 3D FE simulations (Abaqus), and they are well situated between the upper and lower bounds. The present H-models are very easy to use and enable to largely reduce the CAD and mesh preparing work, the memory storage and the computation time.

The future work will be carried out on the other stress states, such as the bending, in plan and transverse shearing, torsion, with the consideration of the skin effect, honeycomb height effect and coupling effect between the Young's moduli and Poisson coefficients.

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