MICRO-MECHANICAL MODELING OF FIBER PULL-OUT STRESSES IN AN AXISYMMETRIC COMPOSITE SYSTEM

S. Kumar *, H. Alhashmi*

Institute Center for Energy (iEnergy), Department of Mechanical and Materials Engineering, Masdar Institute of Science and Technology, PO Box 54224, Abu Dhabi, UAE.
** Corresponding Author: kshanmugam@masdar.ac.ae* Corresponding Author: halhashmi@masdar.ac.ae*

Keywords: Interphase, Interface, Stress analysis, Variational method, Finite Element Analysis,

Abstract

This paper presents an analytical model for the stress analysis of a three phase fiber-matrix composite system under thermo-mechanical loads. This cylindrical assembly consists of a fiber embedded in a matrix having an interphase between them. Matrix and the interphase are regarded as an isotropic linear elastic material while the fiber is regarded as orthotropic linear elastic continuum. Adopting a stress function approach and incorporating stress continuity conditions at the interfaces as well as exactly satisfying traction-free boundary conditions, stress fields in the entire assembly are expressed in terms of two unknown functions. Governing differential equations of the problem are obtained by using variational method in conjunction with complementary energy principle and are numerically solved. The problem has also been computationally studied using Abaqus FEA. This model enables prediction of pull-out capacity of a single fiber composite system comprising an explicit interphase.

1. Introduction

Composites are an important class of materials which have a wide acceptance in different industries. The composite performance and properties will depend on the properties of the matrix and fiber. During the fabrication process a third constituent, interphase, is formed between the matrix and fiber. The interphases are formed as a result of the chemical and physical interaction of the matrix and fiber. They have distinct properties from the parent materials [5]. They act as the buffer therefore being responsible for the stress transfer between the matrix and fiber [7].

Over the last 50 years there was a big number of continuum mechanics model to describe the stress transfer between the differernt constituent of the composites [4]. Those composites can be considered as three phases system, fiber, matrix, and interphase, with averaged properties [2] [1]. Big number of theoretical and numerical studies related the interphase properties to the overall mechanical behavior of the composite [8]. As a simplification, many micromechanical models regarded the interphase properties to be homogenous [3].

Different modeling techniques can be employed to obtain the stress fields; e.g., shear lag model

[8]. Analytical model was developed based on a variational principle which minimizes the complimentary energy of the composite system [6]. The analytical model was developed for a system of fiber and matrix only, with homogeneous properties. The model is to be further modified to account for the third member. This paper will be comparing the stress fields for a composite with homogenous interphase for the analytical model against the results from the FE model.

2. Axisymetric Model

In this study, we analyze the stress fields in a composite system comprising a single fiber embedded in a matrix with an interphase in between. One end of the fiber is subjected to traction σ_0 and temperature change 'ΔT' and the other end of the matrix is fixed. b is the radius of fiber, c the radius of interphase and d is the radius of matrix. The origin of the coordinate system (r,θ,z) is at the top center, as shown in Fig. 1.

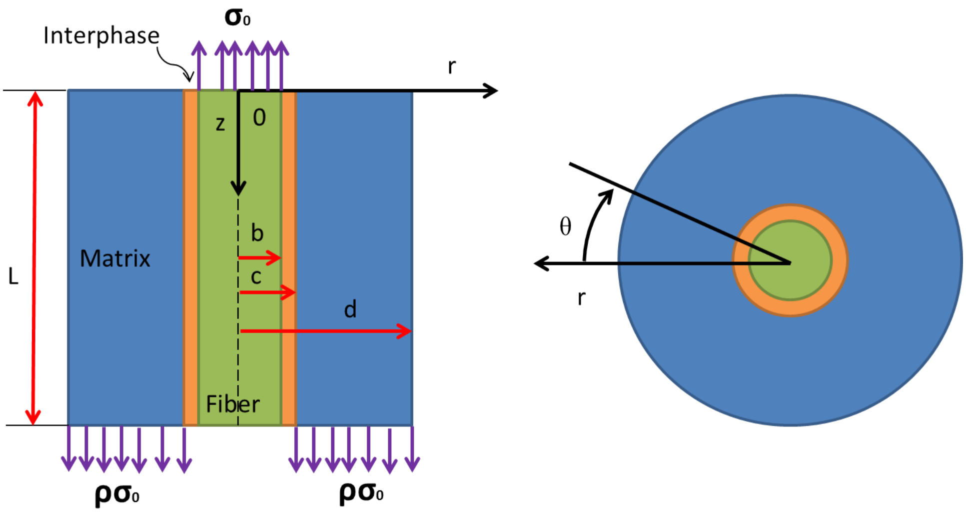

Figure 1: Schematic of the composite system subjected to thermo-mechanical load

2.1. Stress Functions

The analysis is started by assuming stress functions satisfying the axisymmetric equilibrium equations:

$$\frac{\partial \sigma_{zz}^{(i)}}{\partial z} + \frac{\partial \tau_{rz}^{(i)}}{\partial r} + \frac{\tau_{rz}^{(i)}}{r} = 0; \qquad \frac{\partial \sigma_{rr}^{(i)}}{\partial r} + \frac{\partial \tau_{rz}^{(i)}}{\partial z} + \frac{\sigma_{rr}^{(i)} - \sigma_{\theta\theta}^{(i)}}{r} = 0 \tag{1}$$

The stress function is assumed such that

$$\phi_i = f_i(r)g_i(z) + h_i(z) \tag{2}$$

Where i =1 for fiber; =2 for interphase; =3 for matrix The stress components in the fiber, interphase and matrix can be obtained from the following:

$$\sigma_{zz}^{(i)} = \frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_i}{\partial r} \tag{3}$$

$$\tau_{rz}^{(i)} = -\frac{\partial^2 \phi_i}{\partial r \partial z} \tag{4}$$

$$\sigma_{rr}^{(i)} = \sigma_{\theta\theta}^{i} = \frac{\partial^2 \phi_i}{\partial z^2} \tag{5}$$

When substituting the assumed function into the equilibrium equations in (1), the equilibrium equations are indeed satisfied. Now plugging in equation (2) into (3),(4), and (5); we can write the stress fields in the system as follows:

$$\sigma_{zz}^{(i)} = \left(f_i''(r) + \frac{1}{r} f_i'(r) \right) g_i(z) \tag{6}$$

$$\tau_{rz}^{(i)} = -f_i'(r) g_i'(z) \tag{7}$$

$$\sigma_{rr}^{(i)} = \sigma_{\theta\theta}^{(i)} = f_i'(r) g_i''(z) + h_i''(z) \tag{8}$$

Note that there are 9 unknown functions, three for each member of the system. By using the traction-free boundary conditions and the stress continuity at the interface, we can express all the stress components both in fiber and matrix in terms of two unknown functions $g_1(z)$ and $g_3(z)$.

2.2. *Stress Fields in terms of $g_1(z)$ and $g_3(z)$*

Adopting a stress function approach and incorporating stress continuity conditions at the interfaces as well as exactly satisfying traction-free boundary conditions, stress fields in the entire assembly are expressed in terms of two unknown functions

Along with the two unknown functions in the stress fields formulations there are two unknown constants λ_1,and λ_3.

$$\lambda_1 = \frac{1}{g_1(0)} \tag{9}$$

$$\lambda_3 = \frac{1}{g_3(L)} \tag{10}$$

3. Constitutive Models

The axisymmetric constitutive relationships are given in the following matrix. The properties of the fiber are anisotropic but the properties of the matrix and interphase are taken to be isotropic.

$$\begin{bmatrix} \epsilon_{zz}^{(i)} \\ \epsilon_{rr}^{(i)} \\ \epsilon_{\theta\theta}^{(i)} \\ \epsilon_{rz}^{(i)} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_z^{(i)}} & \frac{-\nu_{rz}^{(i)}}{E_r^{(i)}} & \frac{-\nu_{\theta z}^{(i)}}{E_\theta^{(i)}} & 0 \\ \frac{-\nu_{zr}^{(i)}}{E_z^{(i)}} & \frac{1}{E_r^{(i)}} & \frac{-\nu_{\theta r}^{(i)}}{E_\theta^{(i)}} & 0 \\ \frac{-\nu_{z\theta}^{(i)}}{E_z^{(i)}} & \frac{-\nu_{r\theta}^{(i)}}{E_r^{(i)}} & \frac{1}{E_\theta^{(i)}} & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{rz}^{(i)}} \end{bmatrix} \begin{bmatrix} \sigma_{zz}^{(i)} \\ \sigma_{rr}^{(i)} \\ \sigma_{\theta\theta}^{(i)} \\ \tau_{rz}^{(i)} \end{bmatrix} + \begin{bmatrix} \alpha_z & 0 & 0 & 0 \\ 0 & \alpha_r & 0 & 0 \\ 0 & 0 & \alpha_\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta T \tag{11}$$

4. Energy Functional

In order to solve for the two unknown functions g1(z) and g3(z), we can minimize the total complementary energy in the system.

$$\Pi^* = U^* + V^* \tag{12}$$

Where U^* is the complementary strain energy and V^* is the complementary potential energy. Since we have a fixed boundary the complementary potential energy is zero.

The system of the three members is assumed to be linear elastic therefore the complementary strain energy is:

$$U^* = \frac{1}{2} \int_{V_i} \sigma^i_{jk} \epsilon^i_{jk} dV_i; \qquad j,k = r,\theta,z \tag{13}$$

The system is axisymmetric which implies $\tau_{r\theta} = \tau_{\theta z} = 0$ and in cylindrical coordinates $dv = rdrd\theta dz$. Therefore the total complementary energy is:

$$\Pi^* = U^* = \int_0^L \int_0^{2\pi} \int_0^r (\frac{1}{2}\sigma^i_{zz}\epsilon^i_{zz} + \frac{1}{2}\sigma^i_{rr}\epsilon^i_{rr} + \frac{1}{2}\sigma^i_{\theta\theta}\epsilon^i_{\theta\theta} + \tau^i_{rz}\epsilon^i_{rz}) rdrd\theta dz \tag{14}$$

We use the stress fields obtained and the constitutive relationship in equation (14) then integrate with respect to r to obtain, where the constant $A_1, A_2, ..., A_{18}$ are a combination of loads, dimensions, and properties of the composite. :

$$\Pi^* = 2\pi \int_0^L \Psi(z, g_1, g_1', g_1'', g_3, g_3', g_3'', \lambda_1, \lambda_3) dz \tag{15}$$

Where

$$\begin{aligned}\Psi = A_1\lambda_1^2 g_1^2 + A_2\lambda_1^2 g_1 g_1'' + A_3\lambda_1\lambda_3 g_1 g_3 + A_4\lambda_1\lambda_3 g_1 g_3'' \\ +A_5\lambda_1 g_1 + A_6\lambda_1^2 g_1'^2 + A_7\lambda_1\lambda_3 g_1' g_3' + A_8\lambda_1^2 g_1''^2 + A_9\lambda_1\lambda_3 g_3 g_1'' \\ +A_{10}\lambda_1\lambda_3 g_1'' g_3'' + A_{11}\lambda_1 g_1'' + A_{12}\lambda_3^2 g_3^2 + A_{13}\lambda_3 g_3 + A_{14}\lambda_3^2 g_3 g_3'' \\ +A_{15}\lambda_3^2 g_3'^2 + A_{16}\lambda_3^2 g_3''^2 + A_{17}\lambda_3 g_3'' + A_{18}\end{aligned}$$

(16)

5. Solution Procedure

After constructing the functional $\Psi(z, g_1, g_1', g_1'', g_3, g_3', g_3'', \lambda_1, \lambda_3)dz$, the calculus of variation is used to get governing equations. In the following equation (i) will represent 1 and 3 correspond to $g_1(z)$ and $g_3(z)$, respectively.

$$\frac{\partial\Psi}{\partial g_i(z)} - \frac{d}{dz}\frac{\partial\Psi}{\partial g_i'(z)} + \frac{d^2}{dz^2}\frac{\partial\Psi}{\partial g_i''(z)} = 0 \tag{17}$$

The first ODE is obtained from differentiating with respect to $g_1(z)$:

$$\begin{aligned}2A_1\lambda_1^2 g_1 + 2A_2\lambda_1^2 g_1 g_1'' + A_3\lambda_1\lambda_3 g_3 + A_4\lambda_1\lambda_3 g_3'' \\ +A_5\lambda_1 - 2A_6\lambda_1^2 g_1' - A_7\lambda_1\lambda_3 g_3' + 2A_8\lambda_1^2 g_1'''' \\ +A_9\lambda_1\lambda_3 g_3'' + A_{10}\lambda_1\lambda_3 g_3'''' = 0\end{aligned} \tag{18}$$

Similarly, the second ODE is obtained from differentiating with respect to $g_3(z)$.

Solving the unknown λ_1 will require differentiating the functional (16) with respect to λ_1 then setting the equation equal to zero to yield the optimal value of λ_1.

$$\begin{aligned}\int_0^L [2A_1\lambda_1 g_1^2 + 2A_2\lambda_1 g_1 g_1'' + A_3\lambda_3 g_1 g_3 + A_4\lambda_3 g_1 g_3'' \\ +A_5 g_1 + 2A_6\lambda_1 g_1'^2 + A_7\lambda_3 g_1' g_3' + 2A_8\lambda_1 g_1''^2 + A_9\lambda_3 g_3 g_1'' \\ +A_1 0\lambda_3 g_1'' g_3'' + A_1 1 g_1'']dz = 0\end{aligned} \tag{19}$$

Similar procedure is followed for λ_3.

There are eight boundary conditions needed to solve the two ODEs and they are:

$$g_1(0) = \frac{1}{\lambda_1}; g_1(L) = 0; g_1'(0) = 0; g_1'(L) = 0; \tag{20}$$

	Fiber	**Interphase**	**Matrix**
$E_r(MPa)$	5,000	3,000	1,000
$E_\theta(MPa)$	5,000	3,000	1,000
$E_z(MPa)$	10,000	3,000	1,000
$\nu_{r\theta}$	0.4	0.35	0.3
$\nu_{\theta z}$	0.3	0.35	0.3
ν_{rz}	0.3	0.35	0.3
$G_{rz}(MPa)$	1,923	1,111	385
$\alpha_r(1/C^0)$	$26e{-}6$	$51e{-}6$	$76e{-}6$
$\alpha_\theta(1/C^0)$	$26e{-}6$	$51e{-}6$	$76e{-}6$
$\alpha_z(1/C^0)$	$-0.26e{-}6$	$51e{-}6$	$76e{-}6$

Table 1: Properties of the composite

$$g_3(0) = 0; g_3(L) = \frac{1}{\lambda_3}; g_3'(0) = 0; g_3'(L) = 0; \tag{21}$$

Now we solve the two ODE along with the two integro-differential equations and we need to solve them simultaneously to obtain functions $g_1(z)$ and $g_3(z)$ and their derivatives. Matlab program bvp4c is used to solve for the two ODEs. Note that at first we would need to approximate the values of λ_1 and λ_3 , since we need them to solve the ODEs, then iterate until their values converge. Using $g_1(z)$ and $g_3(z)$ and their derivatives stress fields in the entire axisymmetric systems can be obtained.

6. Results and Discussion

6.1. Dimensions, loads, and properties

The length of the fiber under consideration is 0.5 mm, similarly the length of the interphase and matrix. The fiber has a radius of 0.005 mm and radius of the matrix, which is assigned the letter d in Fig.(1), is 0.1 mm. The ratio of the interphase thickness to the fiber radius considered is 0.2. The traction, σ_0 , applied the fiber top is 100 MPa. The temperature difference in the system is $\Delta T = -100C^0$.The Properties of the fiber, interphase, and matrix are shown in table (1).

6.2. Analytical and FE model results

In order to verify the analytical model, first we are going to compare it to the results obtained from the FE model. The comparison will take place at the two interfaces, at b and c. The values to be compared are σ_{zz}, τ_{rz}, and σ_{rr}. Note $\sigma_{\theta\theta}$ was not considered because in the analytical formulation it is assumed to equal to σ_{rr} . The mentioned assumption is valid because even in the FE model the values are about the same. From Fig.(2) to Fig.(5) the analytical results and FE model results have similar behavior with a small difference in magnitude.

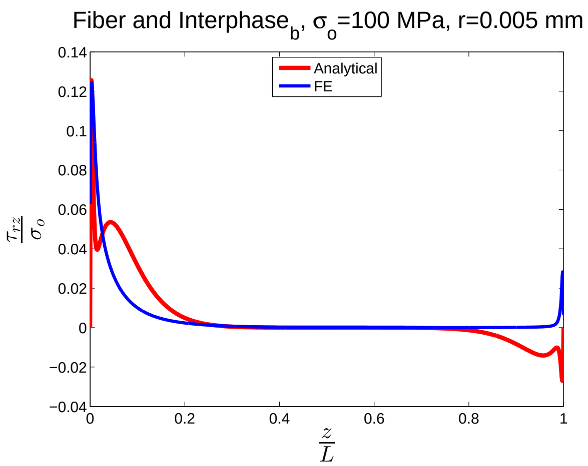

Figure 2: Shear stress along the length of the interface, at b

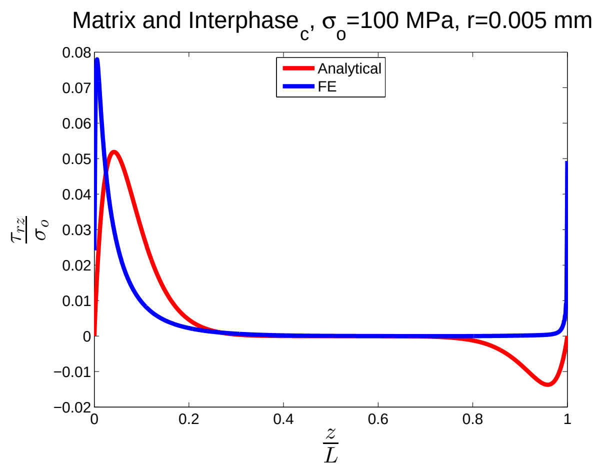

Figure 3: Shear stress along the length of the interface, at c

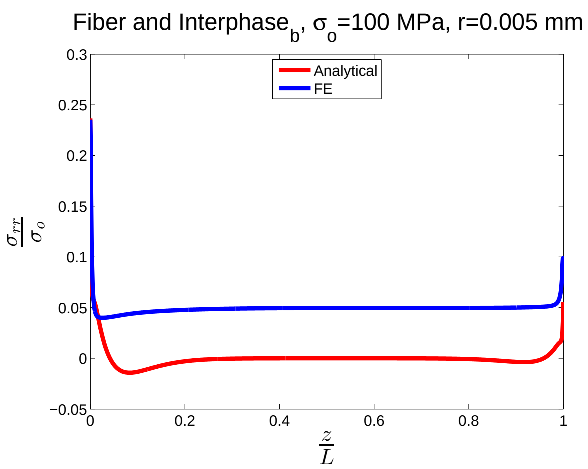

Figure 4: Radial stress along the length of the interface, at b

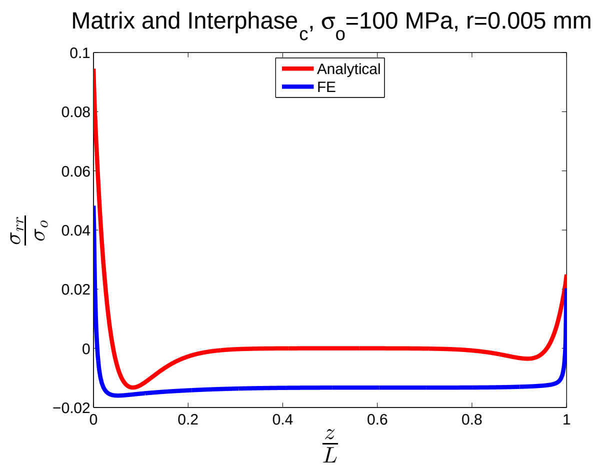

Figure 5: Radial stress along the length of the interface, at c

7. Conclusions

Stress fields in an axisymmetric composite system comprising explicit homogeneous interphase have been analytically obtained by a variational method in conjunction with principle of complementary energy. Interfacial and interphasical shear and radial stress peaks predicted by the analytical model are in good agreement with FE results.

References

[1] Narin J A. .. *Comput Mater Sci*, 40:525, 2007.

[2] DiBenedetto AT. .. *Mater Sci Eng*, A302:74, 2001.

[3] Jones FR Hayes SA, Lane R. Fiber/matrix stress transfer through a discrete interphase. part 1-single-fiber model composites. *composites*, Part A 32(3-4):379–89, 2001.

[4] Jancar J. Review of the role of the interphase in the control of composite performance on micro- and nano-length scales. *J Mater Sci*, 43:6747–6757, 2008.

[5] Drzal LT. The interphase in epoxy composites. *Advances in Polymers Science*, 1986.

[6] Yue C. Y. Quek M. Y. Axisymmetric stress distribution in the single filament pull-out test. *Materials Science and Engineering*, A 189:105–16, 1994.

[7] Pitchumani R. Long-term durability of polymeric matrix composites. *Interphases in Composites*, 2012.

[8] Chen P Yao Y, Chen S. The effect of a graded interphase on the mechanism of stress transfer in a fiber-reinforced composite. *Mechanics of Materials*, 58:35–54, 2013.