HOMOGENIZATION ANALYSIS OF 3-D COMPOSITES USING FINITE THICKNESS UNIT-CELL MODEL

M.R.E. Nasution[a*], N. Watanabe[a], A. Kondo[a,b], A. Yudhanto[c]

[a]*Department of Aerospace Engineering, Tokyo Metropolitan University, 6-6 Asahigaoka, Hino-shi, Tokyo 191-0065, Japan*
[b]*e-Xtream Engineering, MSC Software Company, 1-23-7 Nishishinjuku, Shinjuku-ku, Tokyo 160-0023, Japan*
[c]*COHMAS Laboratory, Physical Science and Engineering Division, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia*
**nasution-ridlo@ed.tmu.ac.jp*

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Abstract

The complexity of 3-D composites microstructure leads to the difficulty on the analysis. However, the microstructure can be idealized such that possesses a periodic pattern, which enables composites structure to be effectively and efficiently analyzed by using asymptotic expansion homogenization (AEH) method involving representative unit-cell (UC) model. In AEH method, analysis of UC model uses a set of periodic boundary condition (BC) which is usually assumed to be periodic in three-dimension. Nevertheless, composite laminates, especially in aerospace application, are very thin. It should not be considered infinitely in the thickness direction. This paper discusses an improvement in asymptotic analysis of periodic structure by modeling the finite thickness unit-cell model and relieving periodic boundary condition at the top and bottom of unit-cell surfaces to consider the effect of finite thickness in 3-D composites.

1. Introduction

Analysis of 3-D composites structure is oftentimes cumbersome due to the very complex and heterogeneous microstructure. Idealizing the complex microstructure as a periodic structure can be a good approach which enables AEH method [1] to be performed to effectively and efficiently analyze 3-D composites. The idealized UC model and the applied periodic BC play an important role in obtaining the results of AEH method. In this analysis, the UC model is considered in two scales, those are microscopic and macroscopic scales. Guedes and Kikuchi [2] used this method for evaluating the averaged elastic constants and equivalent stress of composite materials. Chung et al [3] employed AEH method for analyzing UC of 2-D plain weave composites. The aforementioned studies exclude the calculation of coefficients of thermal expansion (CTE) and influence of thermal residual stresses. In this analysis, the thermal effect is considered due to its potentiality to affect composite damage behavior [4, 5]. Several AEH studies involving thermo-mechanical properties were conducted by Shabana and Noda [6] and Dasgupta et al [7]. AEH method was also performed in the thermomechanical analysis of 3-D orthogonal interlock composites by Nasution et al [8].

The aforementioned studies employing AEH method employed a UC model which is considered to be repeated infinitely in three-dimension (in-plane and out-of-plane directions). However, 3-D composite laminates, especially in aerospace application, are very thin. It should not be considered infinitely in the thickness direction. The influence of finite thickness in composites structure was suggested by Woo and Whitcomb [9] as a future study. Furthermore, the architectures of 3-D composites are usually not periodic in the thickness direction, which necessitates the employment of a UC model representing the whole thickness of structure to obtain more accurate results. In this paper, an improvement in asymptotic analysis is performed by modeling the finite thickness unit-cell model of 3-D orthogonal interlock composites, and relieving periodic BC at the top and bottom of unit-cell surfaces [10].

2. AEH method

2.1.General concept

An elastic body *Ω*, shown in Fig. 1, is subjected to traction *t* and body forces *f*, where the displacement field is prescribed on Γ_d. The body consists of a large amount of heterogeneous and periodic microstructure which can be represented by a unit-cell (UC). In homogenization analysis, the heterogeneous UC can be viewed from two spatial scales, i.e. macroscopic scale (*x*- coordinate system) and microscopic scales (*y*- coordinate system) as seen in Fig. 2.

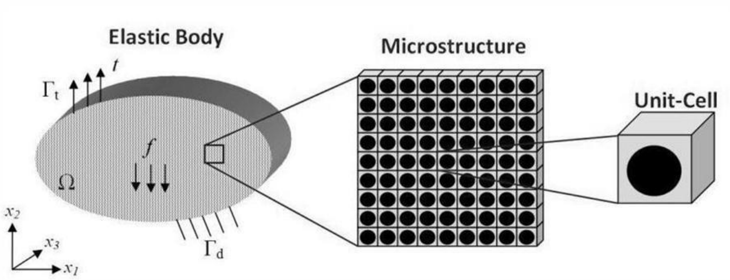

Figure 1. Elastic body with heterogeneous and periodic microstructure.

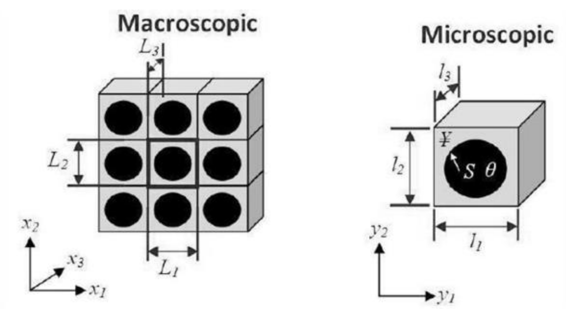

Figure 2. UC viewed from macroscopic and microscopic scales.

The principle of virtual work is employed as a governing equation in this analysis. The mathematical expression in Eq. (1) includes the thermal effect represented in strain term.

$$\int_{\Omega^\varepsilon} E^\varepsilon_{ijkl}\left(\frac{\partial u^\varepsilon_k}{\partial x_l} - \alpha^\varepsilon_{kl}\Delta T\right)\frac{\partial v_i}{\partial x_j} d\Omega = \int_{\Omega^\varepsilon} f_i^{\,\varepsilon} v_i d\Omega + \int_{\Gamma_t} t_i v_i d\Gamma + \int_{S^\varepsilon} p_i^\varepsilon v_i dS \qquad (1)$$

AEH method represents the displacement field by using AE series as follows

$$u_k^{\varepsilon}(\mathbf{x},\mathbf{y}) = u_k^{\varepsilon}(\mathbf{x},\mathbf{y}) + \varepsilon u_k^1(\mathbf{x},\mathbf{y}) + \varepsilon^2 u_k^2(\mathbf{x},\mathbf{y}) + \cdots \tag{2}$$

Microscopic displacement u^1 is obtained by involving solution of variational problem (i.e. characteristic displacements or correctors) [2].

$$u_k^1(\mathbf{x},\mathbf{y}) = -\chi_k^{pq}(\mathbf{x},\mathbf{y}) \frac{\partial u_p^0(\mathbf{x})}{\partial x_q} - \Psi_k(\mathbf{x},\mathbf{y}) \tag{3}$$

where χ and ψ are the elastic and thermal correctors, respectively.
The microstructure variables vary within the unit-cell in both microscopic and macroscopic scales, mathematically represented by periodic vector function g expressed in Eq. (4).

$$g^{\varepsilon}(\mathbf{x}) = g(\mathbf{x},\mathbf{y}) = g(\mathbf{x},\mathbf{y}+\mathbf{Y}) \tag{4}$$

where ε = **x**/**y** and **Y** is the unit-cell dimension.

2.2.Formulation of AEH method with in-plane periodicity

This chapter briefly discusses the formulation of AEH method with only in-plane periodicity wherein the periodicity in the thickness direction is omitted. The detail of formulation can be found in [10]. The existence of only in-plane periodicity necessitates the modeling of through thickness unit-cell. The microstructure variables in thickness direction only vary within the microscopic scale, not in the macroscopic scale. Such kind of variation is represented by

$$g^{\varepsilon}(\mathbf{x}) = g(x_1, x_2, y_1 + Y_1, y_2 + Y_2, y_3) \tag{5}$$

Periodic vector function in Eq. (5) yields the derivatives with respect to macroscopic coordinate x as follows

$$\begin{aligned} \frac{\partial g^{\varepsilon}}{\partial x_1} &= \frac{\partial g}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial g}{\partial y_1} \\ \frac{\partial g^{\varepsilon}}{\partial x_2} &= \frac{\partial g}{\partial x_2} + \frac{1}{\varepsilon}\frac{\partial g}{\partial y_2} \\ \frac{\partial g^{\varepsilon}}{\partial x_3} &= \frac{1}{\varepsilon}\frac{\partial g}{\partial y_3} \end{aligned} \tag{6}$$

Periodic and heterogeneous microstructure is considered as macroscopically homogeneous structure when the limit expressions below exist.

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \int_{\Omega^{\varepsilon}} g^{\varepsilon}(\mathbf{x}) d\Omega &\to \frac{1}{|Y|} \int_{\Omega} \int_{¥} g(\mathbf{x},\mathbf{y}) \, dY d\Omega \\ \lim_{\varepsilon \to 0^+} \varepsilon \int_{S^{\varepsilon}} g^{\varepsilon}(\mathbf{x}) d\Omega &\to \frac{1}{|Y|} \int_{\Omega} \int_{S} g(\mathbf{x},\mathbf{y}) \, dS d\Omega \end{aligned} \tag{7}$$

where $dY = dy_1dy_2dy_3$ and $d\Omega = dx_1dx_2$. Substituting Eqs. (2) and (3) into Eq. (1), and involving Eq. (6) will results in three hierarchical equations based on the order of ε which are solved by taking the limit using expressions (7).

(i). Order of ε^{-2}:

$$\frac{1}{|Y|}\int_{\Omega}\int_{¥} E_{ijkl}\frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial y_j}dYd\Omega = 0 \tag{8}$$

Virtual displacement is arbitrary, and can be a function of either *x*- or *y*- coordinate system. Choosing **v**=**v**(**y**), applying integration by parts and Gauss' divergence theorem, considering the in-plane periodic BC as well as free traction boundary at the top and bottom of unit-cell surfaces yields

$$\frac{1}{|Y|}\int_{\Omega}\left[-\int_{¥}\frac{\partial}{\partial y_j}\left(E_{ijkl}\frac{\partial u_k^0}{\partial y_l}\right)v_i\left(y\right)dY\right]d\Omega = 0 \tag{9}$$

Eq. (9) implies that $u_k^0 = u_k^0\left(x_1, x_2\right)$. This expression asserts that the macroscopic problem is a two-dimensional problem.

(ii). Order of ε^{-1}:

$$\frac{1}{|Y|}\int_{\Omega}\int_{¥} E_{ijkl}\left[\frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial x_j}+\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}-\alpha_{kl}\Delta T\right)\frac{\partial v_i}{\partial y_j}\right]dYd\Omega = \frac{1}{|Y|}\int_{\Omega}\int_{S} p_i v_i dSd\Omega \tag{10}$$

Choosing **v**=**v**(**x**) and considering $u_k^0 = u_k^0\left(x_1, x_2\right)$ implies

$$\int_{S} p_i v_i dS = 0 \tag{11}$$

Choosing **v**=**v**(**y**) and substituting Eq. (3) will obtains two microscopic equilibrium equations as follows
For elastic problem:

$$\int_{¥}\left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i\left(\mathbf{y}\right)}{\partial y_j}dY = 0 \tag{12}$$

For thermal problem:

$$\int_{¥} E_{ijpq}\left(\frac{\partial \psi_p}{\partial y_q} - \alpha_{pq}\Delta T\right)\frac{\partial v_i\left(\mathbf{y}\right)}{\partial y_j}dY = 0 \tag{13}$$

where *i*, *j*, *k*, *p*, *q* = 1, 2, 3 and *l* = 1, 2. It is important to note that the index '*k*' in Eq. (12) is 1 and 2 due to the symmetry of elastic characteristic displacement vector χ, so that there are only three modes of χ, namely χ^{11}, χ^{12}, and χ^{22}.

(iii). Order of ε:

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega}\int_{¥} E_{ijkl}\left[\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}-\alpha_{kl}\Delta T\right)\frac{\partial v_i}{\partial x_j}+\left(\frac{\partial u_k^1}{\partial x_l}+\frac{\partial u_k^2}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}\right]dYd\Omega=\\&\frac{1}{|Y|}\int_{\Omega}\int_{¥} f_i v_i dYd\Omega+\int_{\Gamma_t} t_i v_i d\Gamma\end{aligned} \tag{14}$$

Choosing **v**=**v**(**x**) and representing the microscopic displacement by Eq. (3) will obtains macroscopic equilibrium equation as follows

$$\int_{\Omega} E_{ijkl}^0 \frac{\partial u_k^0}{\partial x_l}\frac{\partial v_i}{\partial x_j}d\Omega=\int_{\Omega}\tau_{ij}\frac{\partial v_i}{\partial x_j}d\Omega+\int_{\Omega}\sigma_{ij}^t\frac{\partial v_i}{\partial x_j}d\Omega+\int_{\Omega} b_i v_i d\Omega+\int_{\Gamma_t} t_i v_i d\Gamma \tag{15}$$

where:
i, k, p, q = 1, 2, 3;   *j, l* = 1, 2
The above formulation implies that omitting the periodic BC in thickness direction will results in homogenized in-plane properties. The homogenized properties are obtained by the following equations.
- Macroscopic homogenized elastic constants:

$$E_{ijkl}^0=\frac{1}{|Y|}\int_{¥}\left(E_{ijkl}-E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)dY \tag{16}$$

- Macroscopic homogenized coefficients of thermal expansion:

$$\alpha_{kl}^0=\frac{1}{\Delta T}\left[S_{ijkl}^0\left(\sigma_{ij}^t+\tau_{ij}\right)\right] \tag{17}$$

where S_{ijkl}^0 is the macroscopic homogenized compliance tensor.

2.3.Periodic boundary conditions

Periodic BCs are employed in the calculation of elastic (χ) and thermal (ψ) correctors by using Eqs. (12) and (13), respectively. The periodic BCs, applied on the unit-cell surfaces, are expressed by Eq. (18) as follows (also valid for thermal correctors by replacing ψ with χ).

$$\begin{aligned}&\chi_k^{pq}(0,y_2,y_3)=\chi_k^{pq}(l_1,y_2,y_3)\\&\chi_k^{pq}(y_1,0,y_3)=\chi_k^{pq}(y_1,l_2,y_3)\\&\chi_k^{pq}(y_1,y_2,0)\neq\chi_k^{pq}(y_1,y_2,l_3)\end{aligned} \tag{18}$$

3. Results and discussion

Numerical analysis is conducted by utilizing finite thickness (FT) UC model (Fig. 3(a)). However, the infinite thickness (IT) model, shown in Fig. 3(b), is also employed to better understand the effect of relieving periodic BC in thickness direction.

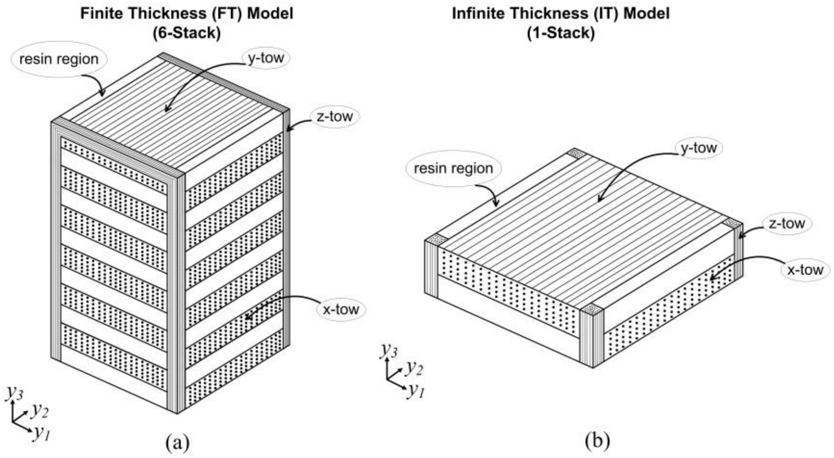

Figure 3. (a) Finite thickness (FT) model, **(b)** Infinite thickness (IT) model.

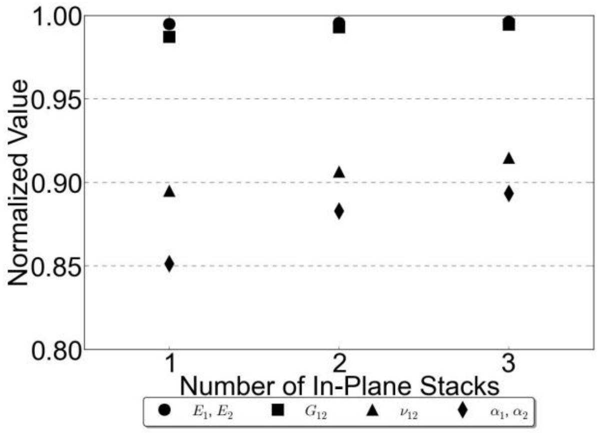

Figure 4. Normalized homogenized properties as influenced by number of in-plane stacks (IT model).

Fig. 4 elucidates the influences of relieving periodic BC in the thickness direction. The influences are evaluated by normalizing the homogenized thermo-mechanical properties obtained by AEH method with 2-D periodicity (i.e. in-plane periodicity) with those obtained by 3-D periodicity. The study is done by increasing the number of in-plane stacks (IT model). The results show that in the case of IT model of 3-D orthogonal interlocked composite, relieving periodic BC is insensitive to the results of elastic and shear moduli. However, it may affect the Poisson's ratio and coefficients of thermal expansion where the increase of number of in-plane stacks tends to reduce the discrepancy with the 3-D periodicity results.

In Fig. 5, the homogenized in-plane thermomechanical properties of FT model calculated by AEH method with 2-D periodicity are normalized by those obtained by 3-D periodicity utilizing both IT and FT models. The values of E_1 and G_{12} normalized to both IT and FT models are found to be insensitive to the increasing of number of in-plane stacks. However, the increasing of in-plane stacks affects the rest of in-plane homogenized thermomechanical properties (i.e. E_2, ν_{12}, α_1 and α_2). This fact shows that the finite thickness UC model is necessary to be employed in the analysis particularly to the model consisting of a few number of in-plane stacks.

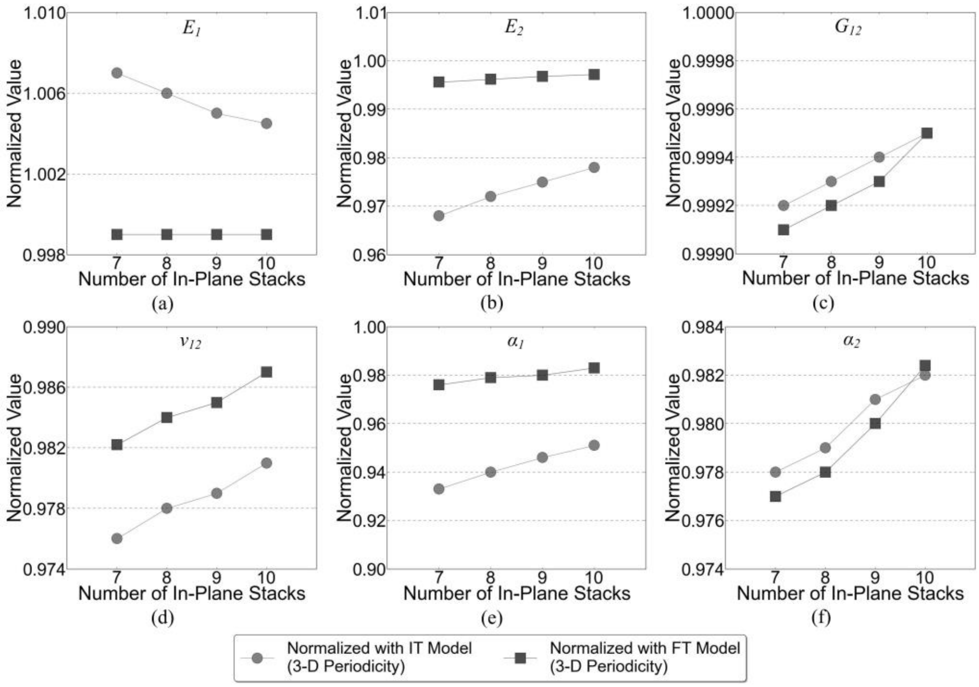

Figure 5. Normalized homogenized thermomechanical properties (FT model).

Properties	E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	$\nu 12$	α_1 (°/C)	α_2 (°/C)
Homogenization (10-Stack FT Model)	59.38	57.55	2.96	0.034	4×10^{-6}	4.12×10^{-6}
Experimental	59.13±1.23	-	-	0.048±0.013	-	-

Table 1. Homogenized properties (10-stack FT model) and experimental results.

Table 1 shows the obtained homogenized thermomechanical properties of 3-D orthogonal interlock composites calculated by 10-stack FT model [10]. The experimental results of E_{11} and ν_{12}, also included in Table 1, are obtained by compression test from Ref. [11]. In the numerical analysis, the idealized FT unit-cell model has a total V_f of 50%, while the V_f of the specimen is 49.5%.

A good agreement between homogenization and experimental results is compared for the elastic modulus E_{11} where the difference of less than 1% is obtained. However, the comparison of Poisson's ratio yields a larger difference (i.e. 28%), despite the fact that the value of homogenization results is still acceptable according to the limit of experimental value. The larger differences can be affected by several factors, among others, the idealization procedures of unit-cell modeling, the sensitivity of ν_{12} to the particular in-plane tows' fiber properties [8] and the influence of strain gages placement [12].

4. Conclusions

Formulation of AEH method for thermomechanical problem has been presented by employing in-plane periodicity. The relieving of periodic boundary condition at the top and bottom of unit cell surfaces consequently results that the macroscopic problem is two-dimensional.

The numerical results show that the relieving of periodicity in the thickness direction may affect the calculated homogenized properties. In addition, the use of finite thickness UC model is found to be important especially when the model consists of a few number of in-plane stacks. In terms of comparison with the experimental results, a good agreement is compared specifically in the result of elastic modulus. A larger difference is found for Poisson's ratio result in spite of the considered acceptable differences according to the limit of experimental value.

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