NUMERICAL MODELING OF STRAIN LOCALIZATION IN FIBER REINFORCED COMPOSITES

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Abstract
This study investigates a numerical model for addressing strain localization appearing in fiber reinforced composite materials. Firstly numerical simulation of localization phenomenon in composite materials is conducted to understand how this deformation phenomenon appears in the material. Secondly mathematical equations expressing deformation of composite materials are compiled, and thirdly the correspondence between mathematical equations and actual deformation phenomenon in composite materials are considered. It is indicated that onset of arbitrariness in solution of equations expressing the deformation of composite materials is closely related with the initiation of compressive failure of composite materials and thus the initiation of narrow localized band in the materials.

1. Introduction

It is well known that many natural phenomena have their mathematical governing equations such as fluids and electro-magnetics, and those governing equations are closely related with the actually observed natural phenomena. The deformation phenomena of composite materials could be also closely related with the mathematical governing equations for deformation of composite materials. This study investigates the relationship between the deformation phenomena of composite materials in strain localization and the mathematical governing equations expressing the deformation of composite materials.

2. Numerical simulation of localization phenomenon in composite materials

Firstly the numerical simulation of localization phenomenon in composite materials is conducted. Finite element method is used to simulate the localization phenomenon. Figure 1 shows the numerical model of this analysis. The white and gray elements in Fig. 1 represent fibers and matrix, respectively. Each fiber and matrix is modeled by two-dimensional plate elements. Fibers are modeled as transversely isotropic elastic material, and matrix is modeled by isotropic elastic-plastic material. CFRP AS4/3501-6 is assumed as the material, and material property shown in Ref. [1] is applied. For nonlinear stress-strain curve of matrix, the data shown in Fig. 2 is applied. The one fiber placed at the center has the initial misalignment as shown in Fig. 1. The initial misalignment of the fiber is introduced using the sine function.

The x coordinate of each node is placed regularly, and the y coordinate of each node is calculated using the sine function. The other fibers are modeled as the straight lines and the fiber axial direction is parallel to the x-direction. Figure 3 and 4 show the simulated results of deformation and stress distribution of the material, respectively. Simulated results show that in the initial state of the loading, the stress concentration occurs in the material around the initial misalignment of fiber, and when the applied load is increased, local areas of matrix around the stress concentration start to yield, and deformation is locally increased. At one

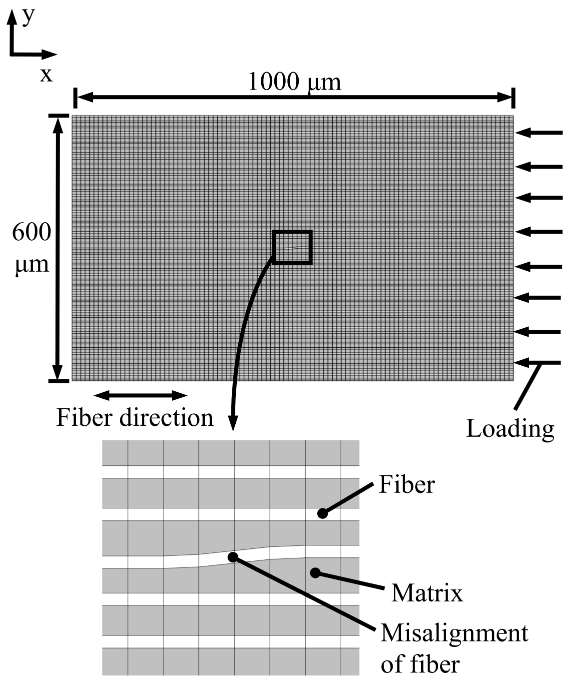

Figure 1. Numerical model of this analysis.

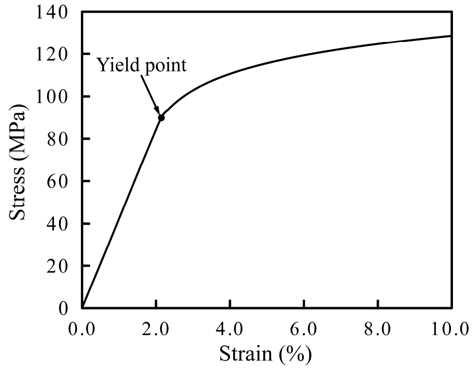

Figure 2. Stress-strain curve of matrix.

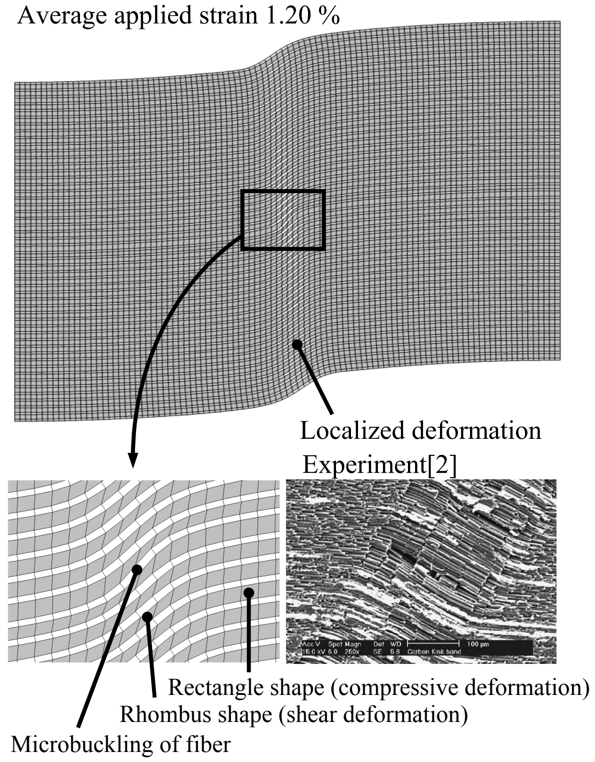

Figure 3. Simulated results of deformation.

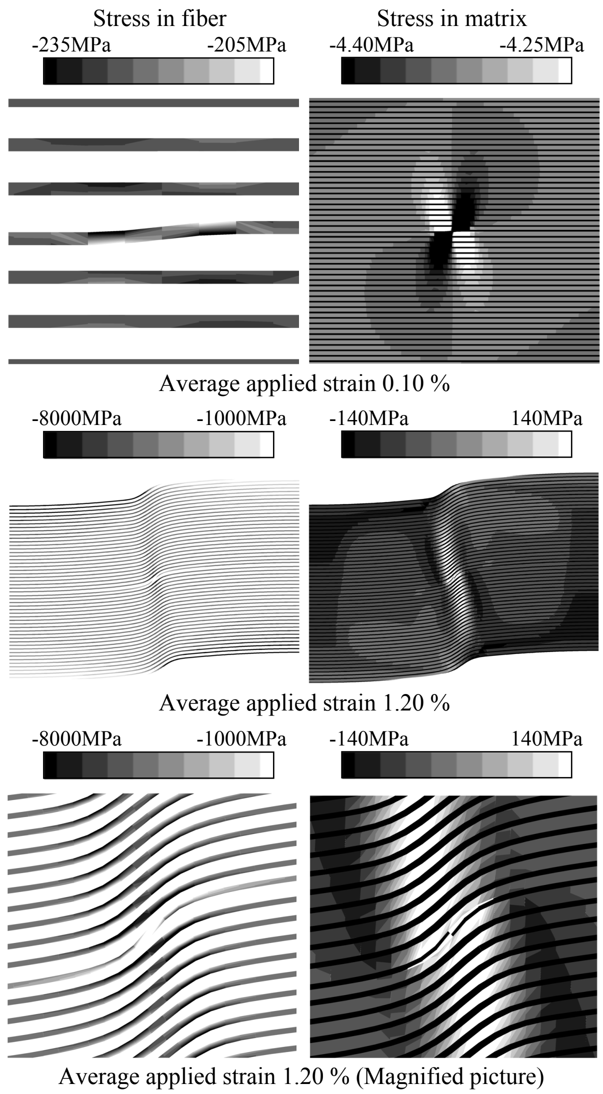

Figure 4. Simulated results of stress distribution.

moment of the loading, a large deformation occurs within a narrow band, and a band of localized deformation develops rapidly. This band of localized deformation passes across the misalignment part of center fiber. As shown in the figures, fibers cause bending deformation, and fiber direction is largely rotated. Matrix causes shear deformation, and the shape of the elements is close to rhombus shape which is rectangle shape in initial state. After the yielding of matrix, the elastic-plastic tangent shear stiffness of matrix significantly reduces, and the shear strain rapidly increases. Then the shear deformation of this part of matrix increases, and due to the shear deformation of the part, the band of localized deformation is formed. The reduction of shear stiffness of matrix is the essential factor in the initiation of the localized deformation of the material.

3. Mathematical equations expressing deformation of composite materials

Secondly the mathematical equations expressing deformation of composite materials are compiled. The equations consist of motion equation and constitutive equation. The motion equation is represented as the following,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial P_{ij}}{\partial X_j} + \rho_0 f_i \tag{1}$$

where ρ_0 is density, t is time, u_i is displacement, X_j is coordinate at reference configuration, P_{ij} is the first Piola-Kirchhoff stress and f_i is external force. The nonlinear stress-strain relation of composite materials is represented by the nonlinear deformation theory shown by Tohgo et al. [3].

$$d\boldsymbol{\sigma} = \boldsymbol{C}_{comp} d\boldsymbol{\varepsilon} = \boldsymbol{C}_m \left\{ \left(1 - V_f\right)\left(\boldsymbol{C}_f - \boldsymbol{C}_m\right)\boldsymbol{S} + \boldsymbol{C}_m \right\}^{-1} \left\{ \left(1 - V_f\right)\left\{\left(\boldsymbol{C}_f - \boldsymbol{C}_m\right)\boldsymbol{S} + \boldsymbol{C}_m\right\} + V_f \boldsymbol{C}_f \right\} d\boldsymbol{\varepsilon} \tag{2}$$

where $d\boldsymbol{\sigma}$ is stress rate, $d\boldsymbol{\varepsilon}$ is strain rate, $\boldsymbol{C}_{comp}$, $\boldsymbol{C}_f$ and $\boldsymbol{C}_m$ are constitutive tensors of composites, fibers and matrix, respectively, V_f is fiber volume fraction and $\boldsymbol{S}$ is Eshelby tensor. Next, the effect of geometrical nonlinearity during the material deformation is considered. Here the constitutive tensor in spacial description is defined in the relation between the second Piola-Kirchhoff stress and the right Cauchy-Green deformation tensor.

$$C_{abcd}^{spa} = \frac{\partial S_{ab}}{\partial C_{cd}^{CG}} \tag{3}$$

where C_{abcd}^{spa} is constitutive tensor in spacial description, S_{ab} is the second Piola-Kirchhoff stress and C_{cd}^{CG} is the right Cauchy-Green deformation tensor. The constitutive tensor in material description is represented by the constitutive tensor in spacial description as follows,

$$C_{ijkl}^{mat} = 2J^{-1} F_{ia} F_{jb} F_{kc} F_{ld} C_{abcd}^{spa} = 2 \frac{1}{J} \frac{\partial x_i}{\partial X_a} \frac{\partial x_j}{\partial X_b} \frac{\partial x_k}{\partial X_c} \frac{\partial x_l}{\partial X_d} C_{abcd}^{spa} \tag{4}$$

where C_{ijkl}^{mat} is constitutive tensor in material description, F_{ia} is deformation gradient, $J = \det F_{ij}$ is Jacobian and x_i is coordinate at present configuration. Cauchy stress is represented by the second Piola-Kirchhoff stress, deformation gradient and Jacobian as follows,

$$\sigma_{ij} = J^{-1} F_{ik} S_{kl} F_{jl} \tag{5}$$

$$\dot{\sigma}_{ij} = J^{-1} \dot{F}_{ik} S_{kl} F_{jl} + J^{-1} F_{ik} \dot{S}_{kl} F_{jl} + J^{-1} F_{ik} S_{kl} \dot{F}_{jl} - \dot{J} J^{-1} F_{ik} S_{kl} F_{jl} \tag{6}$$

where σ_{ij} is Cauchy stress and $\dot{\sigma}_{ij}$ is the material time derivative of Cauchy stress. Here, the time derivative of deformation gradient and Jacobian is

$$\dot{F}_{ij} = L_{ik} F_{kj}, \ \dot{J} = L_{ii} \tag{7}$$

where L_{ik} is velocity gradient. Then

$$\begin{aligned} \dot{\sigma}_{ij} &= J^{-1} L_{im} F_{mk} S_{kl} F_{jl} + J^{-1} F_{ik} \dot{S}_{kl} F_{jl} + J^{-1} F_{ik} S_{kl} F_{ml} L_{jm} - J^{-1} L_{mm} F_{ik} S_{kl} F_{jl} \\ &= J^{-1} F_{ik} \dot{S}_{kl} F_{jl} + L_{ik} \sigma_{kj} + \sigma_{ik} L_{jk} - \sigma_{ij} L_{ll} \end{aligned} \tag{8}$$

where

$$\dot{S}_{kl} = C^{spa}_{klmn} \cdot \dot{C}^{CG}_{mn} = C^{spa}_{klmn} \dot{F}_{om} F_{on} + C^{spa}_{klmn} F_{om} \dot{F}_{on} = C^{spa}_{klmn} L_{op} F_{pm} F_{on} + C^{spa}_{klmn} F_{om} L_{op} F_{pn} \tag{9}$$

$$\begin{aligned} J^{-1} F_{ik} \dot{S}_{kl} F_{jl} &= J^{-1} F_{ik} F_{jl} F_{pm} F_{on} C^{spa}_{klmn} L_{op} + J^{-1} F_{ik} F_{jl} F_{om} F_{pn} C^{spa}_{klmn} L_{op} \\ &= \frac{1}{2} C^{mat}_{ijpo} L_{op} + \frac{1}{2} C^{mat}_{ijop} L_{op} = C^{mat}_{ijkl} \cdot \frac{1}{2} \left(L_{lk} + L_{kl} \right) = C^{mat}_{ijkl} D_{kl} \end{aligned} \tag{10}$$

Therefore

$$\dot{\sigma}_{ij} = C^{mat}_{ijkl} D_{kl} + L_{ik} \sigma_{kj} + \sigma_{ik} L_{jk} - \sigma_{ij} L_{ll} \tag{11}$$

This coinsides with the formulation of Truesdell rate of Cauchy stress. There, here the formulation of finite deformation is based on Truesdell rate of Cauchy stress. Then the rate of the first Piola-Kirchhoff stress is represented as follows,

$$\dot{P}_{ij} = J \frac{\partial X_j}{\partial x_k} \left(\dot{\sigma}_{ik} + \sigma_{ik} L_{ll} - \sigma_{il} L_{kl} \right) = J \frac{\partial X_j}{\partial x_m} \left(C^{mat}_{imkl} D_{kl} + \sigma_{lm} L_{il} \right) = J \frac{\partial X_j}{\partial x_m} \left(C^{mat}_{imkl} + \sigma_{lm} \delta_{ik} \right) \frac{\partial \dot{u}_k}{\partial x_l} \tag{12}$$

where δ_{ik} is Kronecker delta. From Eqs. (1) and (12), a set of equations expressing deformation of composite materials is obtained.

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial P_{ij}}{\partial X_j} + \rho_0 f_i \tag{13}$$

$$\dot{P}_{ij} = J \frac{\partial X_j}{\partial x_m} \left(C^{mat}_{imkl} + \sigma_{lm} \delta_{ik} \right) \frac{\partial \dot{u}_k}{\partial x_l} \tag{14}$$

4. Arbitrariness appearing in solution of equations in deformation of composite materials

Equations (13) and (14) are unified to one differential equation.

$$\rho_0 \frac{\partial^2 \dot{u}_i}{\partial t^2} - \rho_0 \dot{f}_i = \frac{\partial}{\partial X_j}\left(A_{ijkl} \frac{\partial \dot{u}_k}{\partial x_l} \right) \tag{15}$$

Where tensor A_{ijkl} is

$$A_{ijkl} = J \frac{\partial X_j}{\partial x_m}\left(C^{mat}_{imkl} + \sigma_{lm}\delta_{ik} \right) \tag{16}$$

Equation (15) plays a role of governing equation in the deformation of composite materials. When the reference configuration is taken at the moment of the present time, and in the place where the external force doesn't act, Eq. (15) becomes as follows,

$$\rho \frac{\partial^2 \dot{u}_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(A_{ijkl} \frac{\partial \dot{u}_k}{\partial x_l} \right) \tag{17}$$

Here, we conduct the transformation of coordinate system for this equation. Firstly each variable is transformed as the following in the transformation of coordinate system.

$$dx_i = \frac{\partial x_i}{\partial x'_a} dx'_a ,\ \dot{u}_i = \frac{\partial x_i}{\partial x'_a} \dot{u}'_a ,\ \frac{\partial}{\partial x_l} = \frac{\partial x'_d}{\partial x_l} \frac{\partial}{\partial x'_d}$$

$$L_{kl} = \frac{\partial x_k}{\partial x'_c} \frac{\partial x'_d}{\partial x_l} \frac{\partial u'_c}{\partial x'_d} = \frac{\partial x_k}{\partial x'_c} \frac{\partial x'_d}{\partial x_l} L'_{cd} ,\ \sigma_{ij} = \frac{\partial x_i}{\partial x'_a} \frac{\partial x_j}{\partial x'_b} \sigma'_{ab} ,\ A_{ijkl} = \frac{\partial x_i}{\partial x'_a} \frac{\partial x_j}{\partial x'_b} \frac{\partial x'_c}{\partial x_k} \frac{\partial x_l}{\partial x'_d} A'_{abcd} \tag{18}$$

where x'_a is the coordinate system after the transformation. Then Eq. (17) is transformed as follows,

$$\rho \frac{\partial^2 \dot{u}'_a}{\partial t^2} = \frac{\partial}{\partial x'_b}\left(A'_{abcd} \frac{\partial \dot{u}'_c}{\partial x'_d} \right) \tag{19}$$

Commonly the governing equations for natural phenomena do not change their form in the coordinate transformation. Next, when the deformation is locally isotropic in 2' and 3' direction, $\partial/\partial x'_2$ and $\partial/\partial x'_3$ are equal to zero, and when the deformation is quasi-static, $\partial/\partial t$ becomes equal to zero, which corresponds with the case when inertia term is infinitesimal, then Eq. (19) becomes as follows,

$$\frac{\partial}{\partial x'_1}\left(A'_{a1c1} \frac{\partial \dot{u}'_c}{\partial x'_1} \right) = 0 \tag{20}$$

Here, the eigenvalue problem of the tensor A'_{a1c1} is considered. Using the eigenvalue λ' and the eigenvector v'_c of the tensor A'_{a1c1}, the eigenvalue problem is represented as

$$A'_{a1c1} v'_c = \lambda' v'_c \tag{21}$$

When the tensor A'_{a1c1} has zero eigenvalues, Eq. (21) becomes as follows,

$$A'_{a1c1} v'_c = 0 \tag{22}$$

Multiplying the arbitrary function $\phi'(x'_1)$,

$$A'_{a1c1} v'_c \phi'(x'_1) = 0 \tag{23}$$

Taking the partial differenciation of x'_1

$$A'_{a1c1} v'_c \frac{\partial}{\partial x'_1} \phi'(x'_1) = 0 \tag{24}$$

This equation means that $\dot{u}'_c = v'_c \phi'(x'_1)$ is one of the solution of Eq. (20). Since $\dot{u}'_c = v'_c \phi'(x'_1)$ is the solution of Eq. (20) for arbitrary function $\phi'(x'_1)$, Eq. (20) have multiple solutions, or the arbitrariness appears in the solution of Eq. (20). This case causes when the tensor A'_{a1c1} has zero eigenvalues. When the tensor A'_{a1c1} has zero eigenvalues, the determinant of A'_{a1c1} becomes zero,

$$\det(A'_{a1c1}) = 0 \tag{25}$$

From Eq. (18), the tensor A'_{a1c1} is represented by the original coordinate system of tensor A_{ijkl}.

$$A'_{a1c1} = A_{ijkl} \frac{\partial x'_a}{\partial x_i} \frac{\partial x'_1}{\partial x_j} \frac{\partial x_k}{\partial x'_c} \frac{\partial x'_1}{\partial x_l} \tag{26}$$

Here, we introduce two tensors n_j and J_{ai} which express the coordinate transformation.

$$n_j = \frac{\partial x'_1}{\partial x_j}, \quad J_{ai} = \frac{\partial x'_a}{\partial x_i} \tag{27}$$

Then Eq. (25) becomes as follows,

$$\det(A'_{a1c1}) = \det\left(A_{ijkl} n_j n_l J_{ai} J_{ck}^{-1}\right) = \det\left(A_{ijkl} n_j n_l\right) \cdot \det(J_{ai}) \cdot \det\left(J_{ck}^{-1}\right) = 0 \tag{28}$$

Since $\det(J_{ai}) \neq 0$,

$$\det\left(A_{ijkl} n_j n_l\right) = 0 \tag{29}$$

As the conclusion of this analysis, when Eq. (29) is satisfied, the arbitrariness appears in the solution of Eq. (20) which is a specific case of the governing equations for the deformation of composite materials. The equation (29) is considered as the initiation condition of arbitrariness in the solution of the equations for the deformation of composite materials. This is interesting because in structural mechanics it is well recognized that the buckling of the structures is represented by a condition where the determinant of the stiffness matrix of the structures is equal to zero.

$$\det[K] = 0 \tag{30}$$

where $[K]$ is the stiffness matrix. There is a significant similarity in between Eq. (29) and Eq. (30). In the case of Eq. (30), at the time when the equation has equality, the structural instability or the buckling phenomena appear in the structures, and the material and geometrical nonlinearity of the stiffness matrix play important roles in these instability or the buckling. In the case of Eq. (29), when the equation has equality, the material instability or the microbuckling phenomena appear in the materials, and the material nonlinearity including the effect of matrix nonlinear stress-strain relation and geometrical nonlinearity including the effect of fiber misalignment play important roles in these instability or the microbuckling. In addition, from Eq. (16), Eq. (29) also becomes as follows,

$$\det\left(C_{ijkl}^{mat} n_j n_l + \sigma_{jl} \delta_{ik} n_j n_l\right) = 0 \tag{31}$$

The first term of this equation depends on the constitutive tensor of the material, including the elastic and plastic property of the material. It is also related with the material nonlinear effect. The second term of the equation depends on the multi-axial stresses. It is related with the geometrical nonlinear effect. The equation indicates that the appearance of arbitrariness is related with the material property and the multi-axial stresses. The angle of microbuckling is able to affect through the variable n_j, but the width of the band of the microbuckling possibly does not affect the arbitrariness condition. It is also notable that due to the nonlinearity including the material and geometrical nonlinearity, the arbitrariness is able to appear, it indicates that the fact that the governing equations for the deformation of composite materials are nonlinear equations is essential for the appearance of arbitrariness. When we put the tensor $A_{ijkl} n_j n_l$ as a_{ik}, the determinant of Eq. (29) is explicitly represented in two-dimensional as the following,

$$\det a_{ik} = a_{11} a_{22} - a_{12} a_{21} = 0 \tag{32}$$

In fiber reinforced composite materials, commonly the elastic modulus in fiber axial direction has much higher value than the value of transverse direction and stress value, and because of this, C_{1111}^{mat} has much higher value than the other components of constitutive tensor C_{ijkl}^{mat} and the components of stress tensor σ_{ij}, that is $C_{1111}^{mat} >> C_{ijkl}^{mat}$, σ_{ij} $\left(C_{ijkl}^{mat} \neq C_{1111}^{mat}\right)$. Since only A_{1111} and a_{11} includes C_{1111}^{mat}, $A_{1111} >> A_{ijkl}$ $\left(A_{ijkl} \neq A_{1111}\right)$ and $a_{11} >> a_{ik}$ $\left(a_{ik} \neq a_{11}\right)$. Thus the equation becomes,

$$a_{22} = \frac{a_{12} a_{21}}{a_{11}} \approx 0 \tag{33}$$

Here the vector n_j is represented using an angle β as follows,

$$n_j = \frac{\partial x_1'}{\partial x_j} = \left(\cos\beta \quad \sin\beta\right) \tag{34}$$

Then a_{22} is represented as follows,

$$\begin{aligned} a_{22} &= A_{2j2l} n_j n_l = C_{2j2l}^{mat} n_j n_l + \sigma_{jl} n_j n_l \\ &= \left(C_{2121}^{mat} + \sigma_{11}\right)\cos^2\beta + \left(C_{2122}^{mat} + C_{2221}^{mat} + 2\sigma_{12}\right)\cos\beta\sin\beta + \left(C_{2222}^{mat} + \sigma_{22}\right)\sin^2\beta \approx 0 \end{aligned} \tag{35}$$

From this equation,

$$-\sigma_{11} \approx C_{2121}^{mat} + \left(C_{2222}^{mat} + \sigma_{22}\right)\tan^2\beta + \left(C_{2122}^{mat} + C_{2221}^{mat} + 2\sigma_{12}\right)\tan\beta \tag{36}$$

$-\sigma_{11}$ is the value of applied compressive stress to the material in longitudinal direction. When this applied stress reaches the value of right hand side of Eq. (36), the determinant of Eq. (29) becomes equal to zero, and the arbitrariness is allowed to appear, which means the instability appears in the material and microbuckling is able to occur in the actual situations. The value of $-\sigma_{11}$ at the time of being equal to right hand side of Eq. (36) is considered as the critical compressive stress σ_{cr} or the buckling stress in microbuckling.

$$\sigma_{cr} \approx C_{2121}^{mat} + \left(C_{2222}^{mat} + \sigma_{22}\right)\tan^2\beta + \left(C_{2122}^{mat} + C_{2221}^{mat} + 2\sigma_{12}\right)\tan\beta \tag{37}$$

Using elastic-plastic tangent shear modulus G_{LT}^{ep}, transverse tangent modulus E_T^{ep}, in-plane Poisson's ratio ν_{12} and ν_{21} and shear stress τ_{12}, the equation becomes as follows,

$$\sigma_{cr} \approx G_{LT}^{ep} + \left(\frac{1}{1-\nu_{12}\nu_{21}} E_T^{ep} + \sigma_{22}\right)\tan^2\beta + \left(C_{2122}^{mat} + C_{2221}^{mat} + 2\tau_{12}\right)\tan\beta \tag{38}$$

In the case of uniaxial compression and if C_{2122}^{mat} and C_{2221}^{mat} are close to zero, the compressive strength is approximately represented as follows,

$$\sigma_{cr} \approx G_{LT}^{ep} + \frac{1}{1-\nu_{12}\nu_{21}} E_T^{ep} \tan^2\beta \tag{39}$$

Equation (39) corresponds with the expression given by Budiansky [4].

5. Conclusions

The relationship between the deformation phenomena of composite materials in strain localization and the mathematical governing equations expressing the deformation of composite materials is investigated. There exists a state where arbitrariness appears in solution of equations expressing deformation of composite materials, and it is indicated that onset of arbitrariness in solution of equations expressing the deformation of composite materials is closely related with the initiation of compressive failure of composite materials and thus the initiation of narrow localized band in the materials.

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