CONSTITUTIVE COMPATIBILITY BASED ELASTIC PARAMETER IDENTIFICATION WITH A DOMAIN DECOMPOSITION APPROACH

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Abstract
We review here the constitutive compatibility method for the identification of local elastic parameters based on full-field measurements. In this method, the statically admissible stress field that can be related through the known constitutive symmetry to the kinematic observations is sought through minimization of an objective function which measures the violation of constitutive compatibility. After this stress reconstruction, the local material parameters are identified with the given kinematic observations using the constitutive equation. The method has been adapted to solve larger identification problems using a domain decomposition technique which allows for reduced computational load and higher identification accuracy within subdomains.

1. Introduction

In this article, we illustrate the capabilities of the constitutive compatibility method (CCM) which has been recently proposed [1] and which has been further developed in a domain decomposition framework. The CCM is very well suited for use in cases where the material properties are assumed spatially non-uniform (i.e. can vary point wise due to underlying microstructure). The advantages of the method include: (1) an uncoupling of the stress from the material parameters in the identification process for isotropic materials; the stress field is first reconstructed and the parameters are identified in a later stage (2) if the solution is not unique, the CCM provides a family of stress fields (3) the regularization of the inverse problem can be controlled through the definition of the stress search space (4) the method naturally fits in a domain decomposition type strategy to allow large-data identification problems to be solved through relatively low-cost eigenvalue problems over subdomains.

We present here the CCM and its domain decomposition implementation for the identification of elasticity parameters in heterogeneous media. The domain decomposition strategy relies on a partitioning of the solution into several solutions over subdomains obtained by assuming pure Dirichlet boundary conditions. The solution over each subdomain is composed of families of stress fields, each with a macroscopic part (that exerts traction over the boundary) and microscopic parts (that exerts zero traction over the boundary). The global stress field is then chosen to maximize compatibility of the macroscopic parts over the interfaces and the material properties can be calculated from the reconstructed stress field and the kinematic measurements.

First, the inverse problem and the CCM are presented in Section 2. In Section 3 we describe the multi-scale domain decomposition identification strategy. The multiscale approach is then demonstrated in Section 4 through a large scale identification problem.

2. Constitutive compatibility

2.1. The inverse problem

Suppose Ω is a continuum whose reference configuration occupies the region of space bounded by $\partial\Omega$. The boundary $\partial\Omega$ is made of two sub-boundaries $S_{\underline{\bar{f}}}$ and $S_{\underline{\bar{u}}}$ such that $\partial\Omega = S_{\underline{\bar{f}}} \cup S_{\underline{\bar{u}}}$ and $S_{\underline{\bar{f}}} \cap S_{\underline{\bar{u}}} = \emptyset$. Let the traction field $\underline{\bar{f}}$ be prescribed over $S_{\underline{\bar{f}}}$ and the displacement field $\underline{\bar{u}}$ be prescribed over $S_{\underline{\bar{u}}}$. Also, let $\underline{\underline{\sigma}}$ denote the Cauchy stress tensor, $\underline{u}$ the displacement vector, $\underline{\underline{\varepsilon}}$ the infinitesimal strain tensor and $\underline{n}$ the outward unit normal to $\partial\Omega$. The material behavior is described by the 4^{th} order stiffness tensor field $\underline{\underline{\underline{\underline{K}}}}$.

The solution $(\underline{u}, \underline{\underline{\varepsilon}}, \underline{\underline{\sigma}})$ of the forward problem has to confirm three sets of equations:

- Kinematic admissibility:

$$\underline{\underline{\varepsilon}}(\underline{x}) = \frac{1}{2}(\underline{\underline{\nabla}}\,\underline{u}(\underline{x}) + \underline{\underline{\nabla}}^t\underline{u}(\underline{x})), \forall \underline{x} \in \Omega \quad ; \quad \underline{u} = \underline{\bar{u}}, \forall \underline{x} \in S_{\underline{\bar{u}}} \tag{1}$$

- Static Admissibility:

$$\underline{div}\,\underline{\underline{\sigma}} = \underline{0}, \forall \underline{x} \in \Omega \quad ; \quad \underline{\underline{\sigma}} \cdot \underline{n} = \underline{\bar{f}}, \forall \underline{x} \in S_{\underline{\bar{f}}} \tag{2}$$

- Constitutive Equation:

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{K}}}}(\underline{x}) : \underline{\underline{\varepsilon}} \quad \forall\, \underline{x} \in \Omega \tag{3}$$

We introduce spaces $\mathcal{C}$, $\mathcal{S}$, and $\mathcal{K}$ defined as follows

- The space of kinematically admissible displacement $\mathcal{C}$:

$$\mathcal{C}(\underline{\bar{u}}) = \{\underline{v} \in \mathcal{V}_{\underline{v}} | \quad \underline{v} = \underline{\bar{u}} \quad \forall\, \underline{x} \in S_{\underline{\bar{u}}}\} \tag{4}$$

- The space of statically admissible stress $\mathcal{S}$:

$$\mathcal{S}(\underline{\bar{f}}) = \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} | \quad \underline{div}\,\underline{\underline{\tau}} = \underline{0} \quad \forall\, \underline{x} \in \Omega, \quad \underline{\underline{\tau}} \cdot \underline{n} = \underline{\bar{f}} \quad \forall\, \underline{x} \in S_{\underline{\bar{f}}}\} \tag{5}$$

- The space of thermodynamically admissible stiffness tensor fields $\mathcal{K}$:

$$\mathcal{K} = \{\underline{\underline{\underline{\underline{K}}}} \in \mathcal{V}_{\underline{\underline{\underline{\underline{K}}}}} \mid \quad \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{K}}}} : \underline{\underline{\varepsilon}} \geq 0 \quad \forall \underline{\underline{\varepsilon}} \in \mathcal{V}_{\underline{\underline{\tau}}}, \quad \underline{\underline{\varepsilon}} = \underline{\underline{0}} \Longleftrightarrow \underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{K}}}} : \underline{\underline{\varepsilon}} = 0\} \tag{6}$$

The inverse problem is to estimate the constitutive tensor field $\underline{\underline{\underline{\underline{K}}}}$ based on information about the boundary conditions $\underline{\bar{u}}$ and $\underline{\bar{f}}$ and kinematic field measurements over the whole domain.

2.2. Concept of constitutive compatibility of stresses

The key idea in the CCM is to prescribe the stress field to be compatible to the measured displacement field in the sense of some assumed symmetries of the constitutive operator.

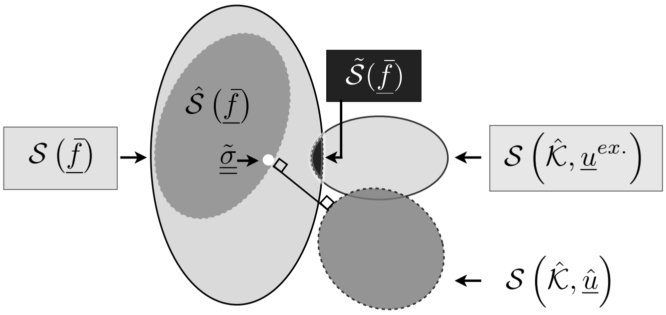

Figure 1. Different spaces involved in the constitutive compatibility method.

Let $\hat{\mathcal{K}}$ $(\subset \mathcal{K})$ be a thermodynamically admissible material stiffness tensor space with a specific material symmetry. We denote the space of stresses that can result from the exact displacement field $\underline{u}^{ex}$ through this material symmetry class as the constitutively compatible stress space, $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{u}^{ex}\right)$,:

$$\mathcal{S}\left(\hat{\mathcal{K}}, \underline{u}^{ex}\right) = \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} \mid \quad \exists \underline{\underline{\underline{\underline{K}}}} \in \hat{\mathcal{K}}, \; \underline{\underline{\tau}} = \underline{\underline{\underline{\underline{K}}}} : \underline{\underline{\varepsilon}}\left(\underline{u}^{ex}\right)\} \tag{7}$$

A viable solution to the inverse problem should belong simultaneously to both spaces $\mathcal{S}\left(\underline{\bar{f}}\right)$ and $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{u}^{ex}\right)$. We call this intersection, $\tilde{\mathcal{S}}(\underline{\bar{f}})$, the "solution stress space" (see Fig. 1).

In practice, $\underline{\hat{u}}$ is reconstructed from noisy, spatially discrete data obtained from a DIC technique. Thus, $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{\hat{u}}\right)$ is an approximation of $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{u}^{ex}\right)$. Also, we can only search for the stress in a (finite-dimensional) subspace $\hat{\mathcal{S}}\left(\underline{\bar{f}}\right)$ of the (most often infinite-dimensional) space $\mathcal{S}\left(\underline{\bar{f}}\right)$. The goal then, is to find the statically admissible stress field $\underline{\underline{\tilde{\sigma}}} \in \hat{\mathcal{S}}(\underline{\bar{f}})$ with minimal distance to the space of constitutively compatible stresses $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{\hat{u}}\right)$.

2.3. The compatibility condition for linear elastic isotropic behavior

Any stiffness tensor has a unique orthonormed basis of elastic eigenstates $\underline{\underline{P_i}}$ and corresponding moduli of rigidity λ_i such that:

$$\underline{\underline{\underline{K}}} = \sum_{i}^{l} \lambda_i \underline{\underline{P}}_i \otimes \underline{\underline{P}}_i \tag{8}$$

where $\lambda_i \geq 0$ and $\underline{\underline{P}}_i : \underline{\underline{P}}_j = \delta_{ij}$ (δ_{ij} denotes the Kronecker delta, $l = 3$ for 2D problems). Let $\mathcal{K}^{iso}$ represent the space of isotropic linear elastic stiffness tensors. Assuming $\underline{\underline{\underline{K}}} \in \mathcal{K}^{iso}$, its eigen-system has a geometric multiplicity corresponding to two unique moduli of rigidity such that the constitutive relation (Eq. (3)) can be rewritten:

$$\underline{\underline{\sigma}} = \lambda_1 \varepsilon_1 \underline{\underline{P_1}} + \lambda_2 \sum_{i=2}^{l} \varepsilon_i \underline{\underline{P}}_i \tag{9}$$

where ε_i are the strain components when projected over the basis of eigenstates.

Thus the stress state $\underline{\underline{\sigma}}$ lies in the $(\underline{\underline{P}}_1, \sum_{i=2}^{l} \varepsilon_i \underline{\underline{P}}_i)$ hyper-plane which is defined only by the kinematic data. We can define $\underline{\underline{\hat{P}}}^{\perp}\left(\underline{x}\right)$, a tensor field orthogonal to the $(\underline{\underline{P}}_1, \sum_{i=2}^{l} \varepsilon_i \underline{\underline{P}}_i)$ hyper-plane. The solution stress space can then be reformulated as:

$$\tilde{\mathcal{S}}(\underline{\bar{f}}) = \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} \mid \underline{\underline{\tau}} : \underline{\underline{\hat{P}}}^{\perp} = 0, \underline{div}\, \underline{\underline{\tau}} = \underline{0}, \forall\, \underline{x} \in \Omega; \underline{\underline{\tau}} \cdot \underline{n} = \underline{\bar{f}} \;\; \forall\, \underline{x} \in S_{\underline{\bar{f}}}\} \tag{10}$$

where $\underline{\underline{\hat{P}}}^{\perp}$ is dependent on the kinematic data $\underline{\hat{u}}$.

The constitutively compatible stress space $\mathcal{S}\left(\hat{\mathcal{K}}, \underline{\hat{u}}\right)$ can be found by minimizing the global violation of the compatiblity condition:

$$\underline{\underline{\tilde{\sigma}}} = \underset{\underline{\underline{\tau}} \in \hat{\mathcal{S}}(\underline{\bar{f}})}{arg\,min} \quad \int_{\Omega} \left(\underline{\underline{\tau}} : \underline{\underline{\hat{P}}}^{\perp}\right)^2 d\Omega \tag{11}$$

2.3.1. Decomposition of the statically admissible stress space

Let $\underline{\underline{\sigma}}^p$ be a statically admissible (macroscopic) stress field (*i.e.* $\underline{\underline{\sigma}}^p \in \hat{\mathcal{S}}$) that conforms to a chosen projection (i.e. uniform, up to linear, up to quadratic ...) of the boundary traction over $\partial\Omega$. Any statically admissible stress field $\underline{\underline{\sigma}}$ that is admissible to the same projection of boundary tractions can then be written:

$$\forall \underline{\underline{\sigma}} \in \hat{\mathcal{S}} \quad \exists\, \underline{\underline{\sigma}}^o \in \mathcal{S}^o \quad | \quad \underline{\underline{\sigma}} = \underline{\underline{\sigma}}^p + \underline{\underline{\sigma}}^o \tag{12}$$

where $\underline{\underline{\sigma}}^o$ is an homogeneous (microscopic) stress field (*i.e.* with zero traction boundary condition) belonging to the space $\mathcal{S}^o$ defined as:

$$\mathcal{S}^o = \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} \,|\quad \underline{div}\,\underline{\underline{\tau}} = \underline{0} \quad \forall\, \underline{x} \in \Omega, \quad \underline{\underline{\tau}} \cdot \underline{n} = \underline{0} \quad \forall\, \underline{x} \in \partial\Omega\} \tag{13}$$

Defining a basis for $\hat{\mathcal{S}}$ requires the construction of (1) a basis of particular stress fields $\underline{\underline{\sigma}}_i^p$ (exerting traction over the local subdomain boundary) and (2) a basis of homogeneous stress fields $\underline{\underline{\sigma}}_i^o$ (tractionless over the local subdomain boundary). A convenient choice of basis used here (detailed earlier [1]) can represent up to linearly varying tractions. Thus, any stress field in $\hat{\mathcal{S}}$ can be expanded in terms of a linear combination of these basis tensor fields

$$\underline{\underline{\sigma}} = \sum_{k=1}^{K} f_k \underline{\underline{\sigma}}_k^p + \sum_{i=1}^{N} d_i \underline{\underline{\sigma}}_i^o \tag{14}$$

d_i, f_i are constant coefficients of homogeneous and particular basis tensor fields respectively.

2.3.2. Obtaining the stress and identifying the parameters

We can now solve for the stress. The objective function is written:

$$\omega = \int_\Omega \left\{ \left(\sum_{i=1}^{N} d_i \underline{\underline{\sigma}}_i^o + \sum_{k=1}^{K} f_k \underline{\underline{\sigma}}_k^p \right) : \underline{\underline{\hat{P}}}^\perp \right\}^2 d\Omega \tag{15}$$

The minimization of Eq. (15) leads to a non-square system for which we seek the eigenvectors and eigenvalues through the pseudo-inverse. Ideally, only $\underline{\underline{\sigma}}^0$ and $\underline{\underline{\sigma}}^1$ are required to describe the stress state in Ω. In general, the stress state can be written as $\underline{\underline{\tilde{\sigma}}} = \sum_{q=0}^{Q} \kappa^q \underline{\underline{\sigma}}^q$ where Q is the total number of eigenvectors and the multiplicative factors κ^q are unknown.

Given the stress, the material parameters can then be identified as follows:

$$\lambda_1 = \frac{\left(\sum_{q=0}^{Q} \kappa^q \underline{\underline{\sigma}}^q \right) : \underline{\underline{\varepsilon}}_1}{\|\underline{\underline{\varepsilon}}_1\|^2} \quad \text{and} \quad \lambda_2 = \frac{\left(\sum_{q=1}^{Q} \kappa^q \underline{\underline{\sigma}}^q \right) : \underline{\underline{\varepsilon}}_{23}}{\|\underline{\underline{\varepsilon}}_{23}\|^2} \tag{16}$$

3. Domain decomposition using the CCM approach

In the domain decomposition strategy, the monolithic CCM identification problem is solved independently over each subdomain E_i. Here we describe the procedure to enforce the interface compatibility conditions and determine $\kappa^q_{E_i}$ that link the stresses in the subdomains to obtain the stress reconstruction over the entire domain.

The strategy is performed in three steps: (1) partitioning the domain into non-overlapping subdomains E_i, (2) solving the Dirichlet boundary condition identification problem over each subdomain, (3) using the families of stress fields $\underline{\underline{\tilde{\sigma}}}_{E_i}$ resulting from step (2) to solve for the macroscopic projections that ensure global equilibrium of the entire domain and perform identification within the domain.

3.1. Partitioning the observed area into subdomains

We begin by partitioning the domain Ω into subdomains E_i such that $\Omega = \bigcup_{i=1}^{N_E} E_i$. Within each subdomain E_i we have part of the full-field observation ($\underline{\hat{u}}$) and no knowledge about the traction boundary conditions (*i.e.* this is a Dirichlet-like problem). In addition, we define a set of interfaces Γ which are either common boundaries, say $\Gamma_{EE'}$, between adjacent subdomains E and E' or part of the boundary of the domain $\partial\Omega$. These external interfaces are regions of either prescribed Dirichlet conditions ($\Gamma_{\underline{\bar{u}}}$) or of prescribed Neumann conditions ($\Gamma_{\underline{\bar{f}}}$) such that $\cup\Gamma_{\underline{\bar{f}}} = S_{\underline{\bar{f}}}$ and $\cup\Gamma_{\underline{\bar{u}}} = S_{\underline{\bar{u}}}$.

The stress solution to the problem over the domain Ω is built as the superposition of the solutions ($\underline{\underline{\tilde{\sigma}}}_{E_i}$) over each sub-domain such that $\underline{\underline{\tilde{\sigma}}} = \bigcup_{E_i \in \Omega} \underline{\underline{\tilde{\sigma}}}_{E_i}$. The stress fields ($\underline{\underline{\tilde{\sigma}}}_{E_i}$) satisfy

- constitutive compatibility :

$$
\begin{aligned}
\underline{\underline{\tilde{\sigma}}}_{E_i} &\in \tilde{\mathcal{S}}_{E_i} = \mathcal{S}_{E_i} \cap \mathcal{S}_{E_i}\left(\hat{\mathcal{K}}, \underline{\hat{u}}|_{E_i}\right) \\
\tilde{\mathcal{S}}_{E_i} &= \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} \mid \quad \underline{\underline{\tau}} : \underline{\underline{\hat{P}}}^{\perp|E_i} = 0 \text{ and } \underline{\mathit{div}}\, \underline{\underline{\tau}} = \underline{0} \quad \forall\, \underline{x} \in E_i\}
\end{aligned}
\tag{17}
$$

- interface compatibility:

$$
\begin{aligned}
\underline{\underline{\tilde{\sigma}}}_{E} \cdot \underline{n}_E &= -\underline{\underline{\tilde{\sigma}}}_{E'} \cdot \underline{n}_{E'} \quad \forall \underline{x} \in \Gamma_{EE'} \\
\underline{\underline{\tilde{\sigma}}}_{E} \cdot \underline{n}_E &= \underline{\bar{f}} \quad \forall \underline{x} \in \Gamma_{\underline{\bar{f}}}
\end{aligned}
\tag{18}
$$

with $\mathcal{S}_{E_i} = \{\underline{\underline{\tau}} \in \mathcal{V}_{\underline{\underline{\tau}}} \mid \quad \underline{\mathit{div}}\, \underline{\underline{\tau}} = \underline{0} \quad \forall\, \underline{x} \in E_i\}$ a statically admissible stress space, and $\mathcal{S}_{E_i}\left(\hat{\mathcal{K}}, \underline{\hat{u}}|_{E_i}\right)$ a constitutively compatible stress space of the local subdomain.

4. Large-scale problem

The domain decomposition approach gains its utility in cases where the amount of the kinematic data to be processed is prohibitively large. A 10 mm $\times$10 mm domain with Young's modulus distribution given in Fig. 2(a) was subjected to a linearly varying traction in the y direction (with profile $\underline{\underline{\sigma}} \cdot \underline{n} = (0.1y + 0.5) \cdot \underline{n}$) that induced a tensile load in the x direction. Fig. 2(b) shows a finer heterogeneity within the structure. A very fine mesh was used to obtain the displacement field from a displacement based FE analysis. For the identification, the domain was divided into 20×20 equally sized subdomains with 31×31 displacement data points over a regular grid in each subdomain.

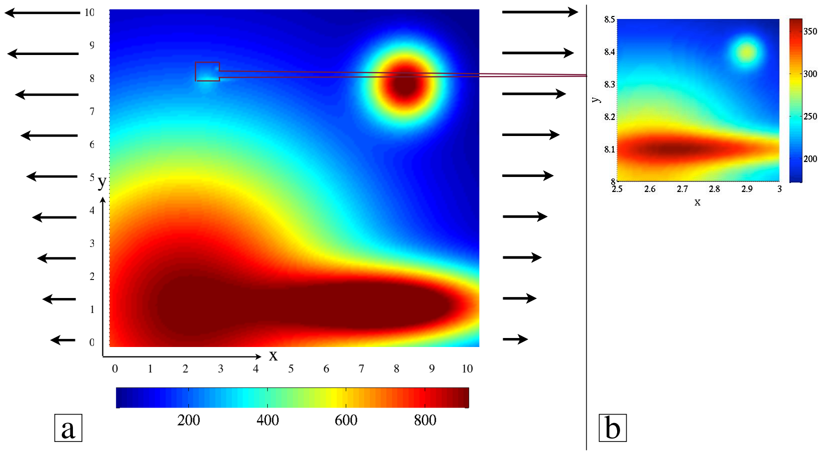

Figure 2. Reference problem with heterogeneous Young's modulus map (in MPa) and linear traction in the x direction (a). A zoom on the finest heterogeneity is shown (b). The Poisson's ratio shares the same distribution with lower and upper limits 0.3 and 0.4 respectively.

In Fig. 3 we see the stress fields reconstructed using the domain decomposition approach. In Fig. 4 the reconstructed stress and the exact stress from FE analysis are given for a region (containing 5×5 subdomains) encompassing the fine heterogeneity within the structure. Fig. 5 shows the Young's modulus estimation resulting from the domain decomposition identification approach. The high error localized at the bottom boundary at $4 \leq x \leq 6$ is present because the load path travels around this region.

5. Concluding Remarks

The domain decomposition CCM method consists of a local stress reconstruction stage over the subdomain through the solving of a Dirichlet problem based on the kinematic data. Then, global compatibility of the stress is achieved by choosing the resulting stress fields that satisfy the global stress boundary conditions and best assure traction continuity over interior interfaces. The domain decomposition approach was shown to capture variations in the stress fields due to a small heterogeneity while also giving a good reconstruction of the stress fields on the global scale. The stress reconstruction then allowed for local identification of the parameters throughout the domain.

References

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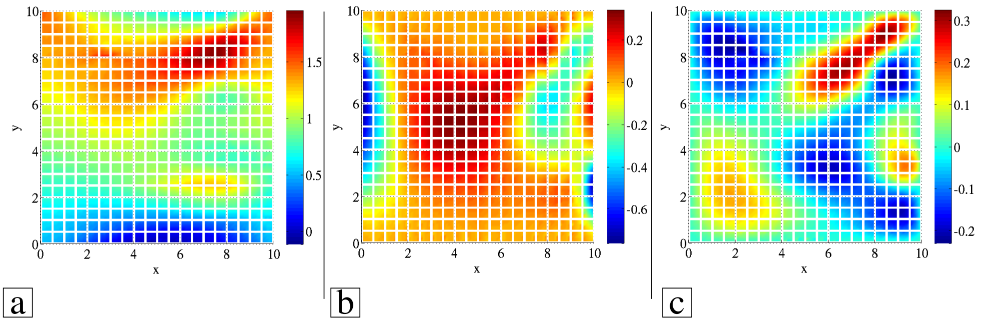

Figure 3. Reconstructed stress fields (a) σ_x, (b) σ_y, (c) σ_{xy} (MPa).

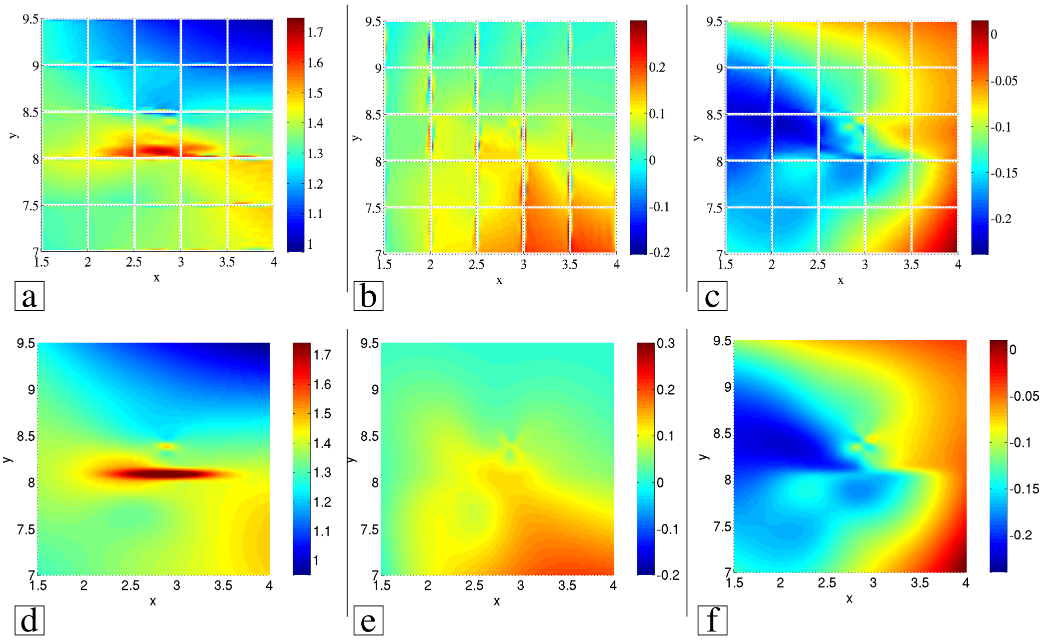

Figure 4. Reconstructed (a) σ_x, (b) σ_y, (c) σ_{xy} and exact stress fields (d) σ_x, (e) σ_y, (f) σ_{xy} (MPa) over $1.4 \leq x \leq 4, 7 \leq y \leq 9.5$.

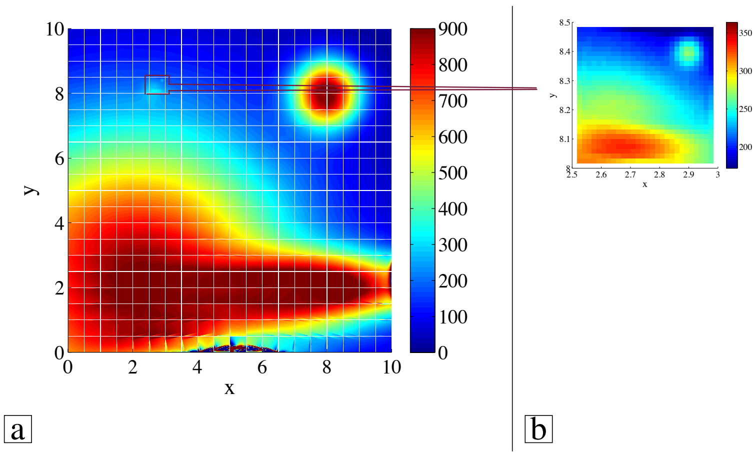

Figure 5. Identified Young's modulus distribution (MPa).