AN APPROXIMATE ANALYTIC SOLUTION FOR THE STRESSES AND DISPLACEMENTS OF THIN-WALLED ORTHOTROPIC BEAMS SUBJECTED TO BENDING

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Abstract

An approximate solution for the stresses and displacements according to the theory of bending with influence of shear is given. It is assumed that the normal stresses in the transverse direction are small compared to the normal stresses in the longitudinal direction so that can be ignored in the stress-strain relations. The solution for the stresses and displacements are given in the analytic form. The unidirectional orthotropic beams with double symmetrical cross-section are considered. The results are compared to the finite element method in several examples.

1. Introduction

The Euler-Bernoulli beam theory as well as the Vlasov's thin-walled beam theory do not take into account shear deformation due to shear forces [1]. The shear deformation effect as well the Poison's effect can be considered by using the methods of theory of elasticity [2,3], but in that case the problem is no longer one-dimensional. Approximate methods which take into account shear effect are developed [4,5,6]. The concept of shear factors first introduced by Timoshenko was used for these analysis [7,8]. Recent approaches to the problem are based on geometric assumptions [9-17] or shear energy relations [6]. Comparisons by numerical examples are given in [18-20].

The solution for the stresses and displacement according to the theory of bending with the shear influence [8,14,15,16,17,21] will be applied for orthotropic thin-walled beams. Pultruded beams are essentially orthotropic, with material principal directions parallel and transversal to the beam longitudinal axis. In that case, the beams can be considered as unidirectional orthotropic beams [22]. Beams with cross-sections with two axes of symmetry are considered. Poisson's effect is ignored, as well as the warping effect, defined by the "non-uniform warping bending theory" [20].

2. Strains and displacements

The displacement of a point $\mathrm{S}(x, s)$ at the middle line in the case of bending of thin-walled beams of open sections with respect to the *xz*-axis of symmetry can be expressed as (Fig. 1)

$$u_S = -\frac{\mathrm{d}w}{\mathrm{d}x} z + u + \int_0^s \gamma_{x\xi}\,\mathrm{d}s = \beta z + u + \int_0^s \gamma_{x\xi}\,\mathrm{d}s\,, \quad \beta = -\mathrm{d}w/\mathrm{d}x \tag{1}$$

where $w = w(x)$ is the displacement in the *z*-direction, i.e. the displacement of the cross-section middle line as a rigid line in the plane of symmetry, $z = z(s)$ is the rectangular coordinate, $u = u(x)$ is the displacement of the cross-section middle line as a rigid line in the *x*-direction, $\gamma_{x\xi} = \gamma_{x\xi}(x,s)$ is the shear strain in the beam middle surface, *s* is the curvilinear coordinate of the middle line, ξ is the tangential axis on the curvilinear coordinate *s*; O*xyz* is the orthogonal coordinate system, where the *y* and *z*-axis are the axes of symmetry; $\beta = \beta(x)$ is the angular displacement of the middle line as rigid line with respect to the *y*-axis, orthogonal to the *z*-axis.

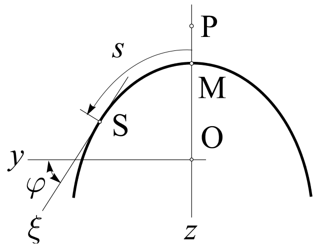

Figure 1. Portion of the cross-section middle line

The displacements can be expressed as

$$w = w_b + w_a\,, \quad u = u_a \tag{2}$$

where where $w_b = w_b(x)$ is the displacement of the cross-sections as plane sections in the *z*-direction, as in the case of the ordinary theory of bending, $w_a = w_a(x)$ is the additional displacement due to shear in the *z*-direction, $u_a = u_a(x)$ is the additional displacement due to shear in the *x*-direction. The angular displacements can be expressed as

$$\beta = \beta_b + \beta_a\,,\ \beta_b = -\mathrm{d}w_b/\mathrm{d}x\,,\ \beta_a = -\mathrm{d}w_a/\mathrm{d}x \tag{3}$$

Thus, the strain in the longitudinal direction can be expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} z - \frac{\mathrm{d}u}{\mathrm{d}x} + \int_0^s \frac{\partial \gamma_{x\xi}}{\partial x}\mathrm{d}s \tag{4}$$

3. Stresses and displacements

Hooke's law for the plane stress condition and unidirectional lamina can be expressed as

$$\sigma_1 = \frac{E_1\varepsilon_1 + E_2\nu_{12}\varepsilon_2}{1-\nu_{21}\nu_{12}}\,,\ \sigma_2 = \frac{E_2\varepsilon_2 + E_2\nu_{12}\varepsilon_1}{1-\nu_{21}\nu_{12}}\,,\ \frac{\nu_{12}}{\nu_{21}} = \frac{E_1}{E_2}\,,\ \tau_{12} = G_{12}\gamma_{12} \tag{5}$$

where σ_1 and σ_2 are the normal stresses in the major (1) and minor (2) directions, respectively; ε_1 and ε_2 are the normal strains; E_1 and E_2 are the moduli of elasticity; ν_{12} is the major Poisson's ratio; τ_{12} is the shear stress and G_{12} is the shear modulus. From Eqs (5)

$$\sigma_1 = E_1\varepsilon_1 f\,, \quad f = \left(1-\nu_{12}\frac{\sigma_2}{\sigma_1}\right)^{-1}. \tag{6}$$

The function *f* can be calculated for the flange of simply supported thin-walled beam with I-section subjected to bending by uniform loads per unit length, for the maximal normal stresses at the beam midspan at the junction of the beam web and the flange [16], using

$$\sigma_2/\sigma_1 = \frac{m^2}{1+\frac{2}{3}\left(\frac{E_1}{G_{12}}-\frac{\nu_{12}}{2}\right)m^2}\,,\ \left(\sigma_2/\sigma_1\right)_{iso.} = \frac{m^2}{1+\frac{2}{3}\left(2+\frac{3}{2}\nu\right)m^2}\,,\ m=\frac{b}{l}\,, \tag{7}$$

for the orthotropic and isotropic material, respectively; where l is the beam length and b the flange breadth. For $E_1 = 53.78$ GPa, $G_{12} = 8.96$ GPa, $\nu_{12} = 0.25$ (glass/epoxy [23]), and $\nu = 0.3$ (steel), the results are given in Tab. 1.

l/b	σ_2/σ_1	$(\sigma_2/\sigma_1)_{iso.}$	f	$f_{iso.}$
3	0.0774	0.0940	1.0197	1.0238
5	0.0346	0.0375	1.0087	1.0105

Table 1. The function *f* , given by (7) and (8), for orthotropic and isotropic material, respectively

The function *f*, as it is shown, is very closed to one, even for extremely low l/b ratios. Thus, the Hooke's law for unidirectional laminas can be expressed as

$$\sigma_x = E_x\varepsilon_x\,,\ \tau_{x\xi} = G\gamma_{x\xi}\,, \tag{8}$$

where $\sigma_x = \sigma_x(x,s) = \sigma_1$, $E_x = E_1$, $\tau_{x\xi} = \tau_{x\xi}(x,s) = \tau_{12}$ and $G_{12} = G$. From Eqs. (4) and (8)

$$\sigma_x = -E_x\frac{\mathrm{d}^2w}{\mathrm{d}x^2}z + E_x\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{E_x}{G}\int_0^s\frac{\partial\tau_{x\xi}}{\partial x}\mathrm{d}s\,. \tag{9}$$

From the equilibrium of a differential portion of the beam wall

$$\tau_{x\xi} = \frac{1}{t}\left[-\int_0^s\frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}s + g(x)\right],\ g = t(\mathrm{M})\cdot\tau_{\xi x}(x,\mathrm{M}) = T_{\mathrm{M}}(x)\,, \tag{10}$$

where $t = t(s)$ is the wall thickness and M is the starting point of the curvilinear coordinate *s*.

If $\partial\tau_{x\xi}/\partial x = const.$, referring to Eqs. (9) and (10)

$$\tau_{x\xi} = \frac{1}{t}\left[T_M + E_x\left(\frac{\mathrm{d}^3w}{\mathrm{d}x^3}S_y(s) - \frac{\mathrm{d}^2u}{\mathrm{d}x^2}A(s)\right)\right],\ S_y(s) = \int_0^s z\,\mathrm{d}A\,,\ A(s) = \int_0^s\mathrm{d}A\,,\ \mathrm{d}A = t\mathrm{d}s \tag{11}$$

$$\tau_{x\xi} = \frac{E_x}{t}\left(-\frac{\mathrm{d}^3w}{\mathrm{d}x^3}S_y^* + \frac{\mathrm{d}^2u}{\mathrm{d}x^2}A^*\right),\ S_y^* = \int_{s^*}z\,\mathrm{d}A^*\,,\ A^* = \int_{s^*}\mathrm{d}A^*\,,\ \mathrm{d}A^* = t\mathrm{d}s^*\,,\ \mathrm{d}s^* = -\mathrm{d}s \tag{12}$$

where $S_y^* = S_y^*(s)$ is the moment of the cut-of portion of area with respect to the *y*-axis, $A^* = A^*(s)$ is the cut-of portion of portion of the beam wall area with respect to *y*-axis, s^* is the curvilinear coordinate of the cut-of portion of the beam wall area, from the free edge, i.e. where $\tau_{x\xi} = 0$.

4. Equilibrium equations

For a portion of the beam wall

$$\sum F_x = \int_L \frac{\partial(\sigma_x t)}{\partial x} \mathrm{d}x\mathrm{d}s = 0, \ \sum F_z = \int_L \frac{\partial(\tau_{x\xi} t)}{\partial x} \sin\varphi \,\mathrm{d}x\,\mathrm{d}s + q_z \mathrm{d}x = 0; \tag{13}$$

$$\int_L \frac{\partial(\tau_{x\xi} t)}{\partial x} \mathrm{d}z + q_z = 0, \ \sin\varphi = \frac{\mathrm{d}z}{\mathrm{d}s}, \tag{14}$$

where $q_z = q_z(x)$ are the forces per unit length acting in the beam plane of symmetry. Referring to Eqs.(9) and (11)

$$-E_x S_y \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + E_x A \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0, \ E_x I_y \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - E_x S_y \frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = q_z; \ A = \int_A \mathrm{d}A, \ I_y = \int_A z^2 \,\mathrm{d}A = 0 \tag{15}$$

where, due to symmetry, $S_y = 0$. Thus

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0, \quad E_x I_y \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q_z \tag{16}$$

5. Internal forces and stresses

Integration of the shear stresses over the cross-sections gives

$$\int_A \tau_{x\xi} \sin\varphi \,\mathrm{d}A = Q_z \tag{17}$$

where $Q_z = Q_z(x)$ is the shear force with respect to the *z*-axis. Substitution of Eq. (12) gives

$$Q_z = -E_x I_y \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}; \quad \int_L S_y^* \,\mathrm{d}z = \int_A z^2 \,\mathrm{d}A = I_y, \ \sin\varphi \,\mathrm{d}A = \sin\varphi \, t\mathrm{d}s = t\mathrm{d}z. \tag{18}$$

Referring to Eqs. (16)

$$\mathrm{d}Q_z / \mathrm{d}x = -q_z, \tag{19}$$

Thus, by substituting Eq.(12) into (18)

$$\tau_{x\xi} = \frac{Q_z S_y^*}{I_y t}. \tag{20}$$

Integration of the normal stresses over the cross-sections gives

$$\int_A \sigma_x \,\mathrm{d}A = 0, \quad M_y = \int_A \sigma_x z \,\mathrm{d}A, \tag{21}$$

where $M_y = M_y(x)$ is the bending moment with respect to the *y*-axis. By substituting Eq.(9) into (21)

$$E_x A \frac{\mathrm{d}u}{\mathrm{d}x} - N^z = 0, \quad M_y = -E_x I_y \frac{\mathrm{d}^2 w}{\mathrm{d}^2 x} - M_y^z, \tag{22}$$

where

$$N^z = \frac{E_x q_z}{G I_y} \int_A \mathrm{d}A \int_0^s \frac{S_y^*}{t} \mathrm{d}s = \frac{E_x q_z}{G I_y} \int_L \frac{A^* S_y^*}{t} \mathrm{d}s, \tag{23}$$

$$M_y^z = \frac{E_x q_z}{GI_y} \int_A z \,\mathrm{d}A \int_0^s \frac{S_y^*}{t} \mathrm{d}s = \frac{E_x q_z}{GI_y} \int_L \left(\frac{S_y^*}{t} \right)^2 \mathrm{d}A \tag{24}$$

Due to symmetry $\int_L \frac{A^* S_y^*}{t} \mathrm{d}s = 0$. Thus, $N^z = 0$ and referring to Eqs. (16), (18) and (22)

$$\frac{\mathrm{d}^2 u}{\mathrm{d}^2 x} = 0, \quad -E_x I_y \frac{\mathrm{d}^3 w}{\mathrm{d}^3 x} = \frac{\mathrm{d}M_y}{\mathrm{d}x} + \frac{\mathrm{d}M_y^z}{\mathrm{d}x} = Q_z, \quad -E_x I_y \frac{\mathrm{d}^4 w}{\mathrm{d}^4 x} = \frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} = \frac{\mathrm{d}Q_z}{\mathrm{d}x} = -q_z \tag{25}$$

The normal stress given by Eq. (9), according to Eqs. (20) and second expression of (25), finally can be expressed as

$$\sigma_x = \frac{M_y}{I_y} z + \frac{M_y^z}{I_y} z - \frac{E_x}{G} \cdot \frac{q_z}{I_y} \int_0^s \frac{S_y^*}{t} \mathrm{d}s \,. \tag{26}$$

The component M_y^z given by Eq. (24) can also be written as

$$M_y^z = \frac{E_x I_y}{GA} \kappa_z q_z, \quad \kappa_z = \frac{A}{I_y^2} \int_A \left(\frac{S_y^*}{t} \right)^2 \mathrm{d}A, \tag{27}$$

where κ_z is the shear factor with respect to the w-displacements. Then, normal stress can be expressed as

$$\sigma_x = \frac{M_y}{I_y} z + \frac{E_x \kappa_z}{GA} q_z z - \frac{E_x}{GI_y} q_z \int_0^s \frac{S_y^*}{t} \mathrm{d}s \,. \tag{28}$$

6. Differential equations with separated displacements

Eqs. (22), according to Eqs. (27), can be expressed as

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M_y}{E_x I_y} - \frac{\kappa_z}{GA} q_z \,. \tag{29}$$

According to Eqs. (2)

$$\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2} = -\frac{M_y}{E_x I_y}, \quad \frac{\mathrm{d}^2 w_a}{\mathrm{d}x^2} = -\frac{k_z}{GA} q_z, \tag{30}$$

Integrating, taking into account Eqs. (3) and (19)

$$u = const., \quad \frac{\mathrm{d}w_a}{\mathrm{d}x} = -\beta_a = \frac{Q_z k_z}{GA} \,. \tag{31}$$

Integration constants are ignored. It is assumed that the angular displacements β_a do not depend on the boundary conditions. Then

$$E_x I_y \frac{\mathrm{d}^3 w_b}{\mathrm{d}x^3} = -\frac{\mathrm{d}M_y}{\mathrm{d}x} = -Q_z, \quad E_x I_y \frac{\mathrm{d}^4 w_b}{\mathrm{d}x^4} = -\frac{\mathrm{d}^2 M_y}{\mathrm{d}x^2} = -\frac{\mathrm{d}Q_z}{\mathrm{d}x} = q_z \,. \tag{32}$$

Integrating the second equation of Eqs. (31) it is obtained that

$$w_a = \frac{\kappa_z}{GA} M_y + C_w . \tag{33}$$

where C_w is integration constant with respect to the w-displacements.

7. Boundary conditions

For starting section A:

$$w_a = 0 , \quad C_w = -\frac{\kappa_z M_{y\mathrm{A}}}{GA} , \quad w = w_b + \frac{\kappa_z \left(M_y - M_{y\mathrm{A}} \right)}{GA} , \tag{34}$$

where $M_{y\mathrm{A}}$ is the bending moment at $x = x_\mathrm{A}$. For simply supported beams, for hinged sections A and B it may be written

$$w\big|_{x=x_{\mathrm{A,B}}} = w_b\big|_{x=x_{\mathrm{A,B}}} = 0 , \quad \mathrm{d}^2 w_b / \mathrm{d}x^2 \big|_{x=x_{\mathrm{A,B}}} = 0 , \quad \left(M_{y\mathrm{A,B}} = 0 \right). \tag{35}$$

For the clamped beams

$$w\big|_{x=x_\mathrm{A}} = w_b\big|_{x=x_\mathrm{A}} = 0 . \qquad \mathrm{d}w_b / \mathrm{d}x_{x=x_\mathrm{A}} = 0 ;$$

$$w\big|_{x=x_\mathrm{B}} = w_b\big|_{x=x_\mathrm{B}} + \frac{\kappa_z E I_y}{GA} \left(-\mathrm{d}^2 w_b / \mathrm{d}x^2 \big|_{x=x_\mathrm{B}} + \mathrm{d}^2 w_b / \mathrm{d}x^2 \big|_{x=x_\mathrm{A}} \right) = 0 , \ \mathrm{d}w_b / \mathrm{d}x_{x=x_\mathrm{B}} = 0 . \tag{36}$$

8. Shear factors

The normal stress at junction of the web and the flange, according to Eq. (30), can be expressed as

$$\sigma_x = \pm \lambda \frac{M_y}{I_y} \cdot \frac{h}{2} , \tag{37}$$

where for beams loaded by uniformly distributed forces per unit length for the beam midspan

$$\lambda = 1 + \frac{8 \kappa_z I_y}{A l^2} \cdot \frac{E_x}{G} \left[1 - \frac{A \left(6 A_1 + A_0 \right) h}{12 I_y t_0 \kappa_z} \right] , \ \lambda = 1 + \frac{24 \kappa_z I_y}{A l^2} \cdot \frac{E_x}{G} \left[1 - \frac{A \left(6 A_1 + A_0 \right) h}{12 I_y t_0 \kappa_z} \right] \tag{38}$$

for simply supported and clamped beams, respectively. The total displacement, according to Eq. (36), can be expressed as

$$w = \eta w_b , \tag{39}$$

where for beams loaded by uniformly distributed forces per unit length for the beam midspan

$$\eta = 1 + \frac{48}{5} \cdot \frac{\kappa_z E_x I_y}{G A l^2} , \quad \eta = 1 + 48 \frac{\kappa_z E_x I_y}{G A l^2} \tag{40}$$

for simply supported beams and clamped beams, respectively.

For simple double symmetrical cross-sections (Fig.2)

$$\kappa_z = \frac{6 \left(2 + \psi \right)^3 \left(30 + 10 \psi + \psi^2 + 5 \psi \rho^2 \right)}{5 \psi \left[12 + \psi \left(8 + \psi \right) \right]^2} , \ I_y = A_1 h^2 \frac{12 + \psi \left(8 + \psi \right)}{12 \left(2 + \psi \right)} , \ A = A_1 \left(2 + \psi \right) \tag{41}$$

where $A_1 = b t_1 , \ A_0 = h t_0 , \ \psi = A_0 / A_1 , \ \rho = b / h$.

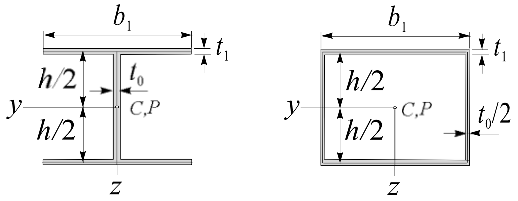

Figure 2. Simple double symmetrical cross-sections

9. Illustrative examples

The factors η and λ, obtained analitically using Eqs. (38) and (40), for simply supported and clamped beams are compared with numerically obtained results by applying the finite element method using ADINA software. The results at the beam midspan are presented in Tab. 2 and Tab. 3. The 9-noded shell elements are used for the FEM analysis of 3D geometry model. The cross-section properties are defined according to Fig. 2: $b = h = 100$ mm, $t_0 = t_1 = 5$ mm. The distributed line load of 1 kN/m is applied to act at neutral axis of the cross-section. The material models are analysed with following properties: for orthotropic material ($E_x = 53.78$ GPa, $G = 8.96$ GPa) and for isotropic material ($E = 210$ GPa, $\nu = 0.3$). The analyses are performed for two different beam lengths ($l/b = 3$ and $l/b = 5$). In the FEM analysis, only one half of the model is analysed using appropriate boundary conditions at the beam midspan and at the starting sections (simple supported and clamped boundary conditions).

l/b	Orthotropic				Isotropic			
	Simply supported		Clamped		Simply supported		Clamped	
	(38)	FEM	(38)	FEM	(38)	FEM	(38)	FEM
3	1.394	1.384	2.181	2.003	1.171	1.176	1.512	1.469
5	1.142	1.142	1.425	1.407	1.061	1.063	1.184	1.168

Table 2. The factors λ, according to (38), for orthotropic and isotropic material, respectively

l/b	Orthotropic				Isotropic			
	Simply supported		Clamped		Simply supported		Clamped	
	(40)	FEM	(40)	FEM	(40)	FEM	(40)	FEM
3	5.207	5.312	22.036	21.710	2.822	2.899	10.112	10.068
5	2.515	2.535	8.573	8.449	1.656	1.676	4.280	4.256

Table 3. The factors η, according to (40), for orthotropic and isotropic material, respectively

10. Conclusion

An analytical solution for bending of thin-walled beams under the influence of shear for double symmetrical cross-sections is given. The shear factors are given in the parametric form in order to compare the shear influence on the beam bending both for orthotropic and isotropic materials. It is shown that the shear influence in the case of unidirectional orthotropic beams is significant, and must be taken into account even in the case of higher beam aspect ratios. Several examples are analyzed in comparison with the finite element method. Excellent agreements of the results are obtained.

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