AN APPROXIMATE ANALYTIC SOLUTION FOR THE STRESSES AND DISPLACEMENTS OF THIN-WALLED ORTHOTROPIC BEAMS SUBJECTED TO TORSION

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Keywords: thin-walled beams, torsion, shear, orthotropic material.

Abstract

An approximate solution for the stresses and displacements according to the theory of torsion with influence of shear is given. It is assumed that the normal stresses in the transverse direction are small compared to the normal stresses in the longitudinal direction that can be ignored in the stress-strain relations. The solution for the stresses and displacements are given in the analytic form. The unidirectional orthotropic beams with double symmetrical cross-section are considered. The results are compared to the finite element method on several examples.

1. Introduction

In classical theories of torsion of thin-walled beams with open section the warping of the cross-sections due to shear is neglected [1-4]. By analogy to advanced theories of bending, in an engineering approach [5-11], the concept of shear factors is considered [12-20]. The solution for the stresses and displacement according to the theory of torsion with influence of shear [12-22] will be applied for orthotropic thin-walled beams. Poltruded beams are orthotropic with principal direction along and normal to the beam longitudinal axis, and can be considered as "unidirectional orthotropic beams" [21-23]. Beams with cross-sections with two axes of symmetry are considered. Poisson's effect is ignored, as well as the shear warping effect, defined by the "non-uniform warping torsion theory" [15].

2. Strains and displacements

The displacement of an arbitrary point $S(x,s)$ of the middle surface of thin-walled beam of open cross-section with one axis of symmetry subjected torsion can be expressed as

$$u_S = -\frac{\mathrm{d}\alpha}{\mathrm{d}x}\omega - \frac{\mathrm{d}v}{\mathrm{d}x}y + \int_0^s \gamma_{x\xi}\,\mathrm{d}s \qquad (1)$$

where $\alpha = \alpha(x)$ is the angle of torsion, i.e. the rotation of the cross-section middle line as a rigid line with respect to a cross-section pole P in the axis of symmetry, $v = v(x)$ is the displacement of the pole P in the y-direction, $y = y(s)$ is orthogonal coordinate, $\gamma_{x\xi} = \gamma_{x\xi}(x,s)$ is the shear strain is the cross-section area in the ξ-direction, s is the curvilinear coordinate of the middle line, ξ is the tangential axis on the curvilinear coordinate *s*; O*xyz* is the orthogonal coordinate system, where the *z*-axis is the axis of symmetry (Fig1);

$$\omega = \int_0^s h_P \, \mathrm{d}s \,, \qquad \mathrm{d}\omega = h_P \mathrm{d}s \,, \tag{2}$$

where $\omega = \omega(s)$ is the sectorial coordinate for the pole P and $h_P = h_P(s)$ is the distance of the tangent through the arbitrary point S at middle line from the pole P.

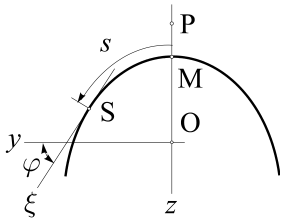

Figure 1. Portion of the cross-section middle line

Here $\omega(s=0)=0$. Eq. (1) may be expressed as

$$u_S = \vartheta\omega - \gamma y + \int_0^s \gamma_{x\xi} \, \mathrm{d}s \,, \quad \vartheta = -\mathrm{d}\alpha/\mathrm{d}x \,, \quad \gamma = \mathrm{d}v/\mathrm{d}x \,; \tag{3}$$

where $\vartheta = \vartheta(x)$ is the relative angular displacement of the middle line as rigid line with respect to the pol P and $\gamma = \gamma(x)$ is angular displacement of the middle line as rigid line with respect to the *z*-axis;

$$\alpha = \alpha_t + \alpha_a \,, \qquad v = v_a \,, \tag{4}$$

where $\alpha_t = \alpha_t(x)$ is the angular displacement of the cross-sections as plane sections with respect to the pole P, as in the case of classical theories of thin-walled beams of open cross-sections, $\alpha_a = \alpha_a(x)$ and $v_a = v_a(x)$ are the additional displacements due to shear;

$$\vartheta = \vartheta_t + \vartheta_a \,, \quad \gamma = \gamma_a \,, \quad \vartheta_t = -\mathrm{d}\alpha_t/\mathrm{d}x \,, \quad \vartheta_a = -\mathrm{d}\alpha_a/\mathrm{d}x \,. \tag{5}$$

The strain in the beam longitudinal direction may then be expressed as

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\omega - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} y + \int_0^s \frac{\partial \gamma_{x\xi}}{\partial x} \mathrm{d}s \tag{6}$$

3. Stresses and displacements

Hooke's law for the plane stress condition and unidirectional lamina can be expressed as

$$\sigma_1 = \frac{E_1\varepsilon_1 + E_2\nu_{12}\varepsilon_2}{1-\nu_{21}\nu_{12}} \,, \ \sigma_2 = \frac{E_2\varepsilon_2 + E_2\nu_{12}\varepsilon_1}{1-\nu_{21}\nu_{12}} \,, \ \frac{\nu_{12}}{\nu_{21}} = \frac{E_1}{E_2} \,, \ \tau_{12} = G_{12}\gamma_{12} \tag{7}$$

where σ_1 and σ_2 are the normal stresses in the major (1) and minor (2) directions, respectively; ε_1 and ε_2 are the normal strains; E_1 and E_2 are the moduli of elasticity; ν_{12} is the major Poisson's ratio; τ_{12} is the shear stress and G_{12} is the shear modulus. From Eqs (6)

$$\sigma_1 = E_1\varepsilon_1 f \,, \quad f = \left(1 - \nu_{12}\frac{\sigma_2}{\sigma_1}\right)^{-1} . \tag{8}$$

The function *f* can be estimated, for example, by using the solution for a strip under self-equilibrated linearly distributed loads along the strip longitudinal edges, for the maximal normal stresses [24]

$$\sigma_2/\sigma_1 = \frac{m^2}{1+\frac{2}{3}\left(\frac{E_1}{G_{12}}-2\nu_{12}\right)m^2},\ (\sigma_2/\sigma_1)_{iso.} = \frac{m^2}{1+\frac{4}{3}m^2},\ m=\frac{b}{l}, \tag{9}$$

for the orthotropic and isotropic material, respectively; where *l* is the strip length and *b* the breadth. For $E_1 = 53.78$ GPa, $G_{12} = 8.96$ GPa, $\nu_{12} = \nu =$ 0.25 (glass/epoxy [24]), the function *f* is calculated and presented in Tab. 1.

l/b	σ_2/σ_1	$(\sigma_2/\sigma_1)_{iso.}$	f	$f_{iso.}$
3	0.0393	0.0909	1.0110	1.0261
5	0.0241	0.0380	1.0068	1.0107

Table 1. The function *f* , given by (7) and (8), for orthotropic and isotropic material, respectively

The function *f*, as it is shown, is very closed to one, even for extremely low l/b ratios. Thus, the Hooke's law for unidirectional laminas can be expressed as

$$\sigma_x = E_x\varepsilon_x,\ \tau_{x\xi} = G\gamma_{x\xi}, \tag{10}$$

where $\sigma_x = \sigma_x(x,s) = \sigma_1$, $E_x = E_1$, $\tau_{x\xi} = \tau_{x\xi}(x,s) = \tau_{12}$ and $G_{12} = G$. From Eqs. (5) and (9)

$$\sigma_x = -E_x\frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2}\omega + E_x\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{E_x}{G}\int_0^s \frac{\partial\tau_{x\xi}}{\partial x}\mathrm{d}s\,. \tag{11}$$

From the equilibrium of a differential portion of the beam wall

$$\tau_{x\xi} = \frac{1}{t}\left[-\int_0^s \frac{\partial(\sigma_x t)}{\partial x}\mathrm{d}s + g(x)\right],\ g = t(\mathrm{M})\cdot\tau_{\xi x}(x,\mathrm{M}) = T_\mathrm{M}(x), \tag{12}$$

where $t = t(s)$ is the wall thickness. If $\partial\tau_{x\xi}/\partial x = const.$, referring to Eqs. (11) and (12)

$$\tau_{x\xi} = \frac{1}{t}\left[T_M + E_x\left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}S_z(s) + \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^2}S_\omega(s)\right)\right],\ S_z(s) = \int_0^s y\,\mathrm{d}A,\ S_\omega(s) = \int_0^s \omega\,\mathrm{d}A,\ \mathrm{d}A = t\mathrm{d}s \tag{13}$$

$$\tau_{x\xi} = -\frac{E_x}{t}\left(\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}S_z^* + \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3}S_\omega^*\right),\ S_z^* = \int_{s^*} y\,\mathrm{d}A^*,\ S_\omega^*(s) = \int_0^s \omega\,\mathrm{d}A^*,\ \mathrm{d}A^* = t\mathrm{d}s^*,\ \mathrm{d}s^* = -\mathrm{d}s \tag{14}$$

where $S_z^* = S_z^*(s)$ is the moment of the cut-of portion of area with respect to the z -axis, $S_\omega^* = S_\omega^*(s)$ is the moment of the cut-of portion of area with respect to the sectorial coordinate ω, s^* is the curvilinear coordinate of the cut-of portion of the beam wall area, from the free edge, i.e. where $\tau_{x\xi} = 0$. The St. Venant shear stress component $\tau_{x\xi}^V = \tau_{x\xi}^V(x,s)$ may be included as

$$\tau_{x\xi}^V = \frac{M_t}{I_t}\eta,\quad M_t = GI_t\frac{\mathrm{d}\alpha_t}{\mathrm{d}x} = -GI_t\theta_t,\quad I_t = \frac{1}{3}\int_L t^3\mathrm{d}s\ , \tag{15}$$

where $M_t = M_t(x)$, and η is the axis orthogonal to the ξ -axis.

4. Equilibrium equations

For a portion of the beam wall

$$\sum F_x = \int_L \frac{\partial(\sigma_x t)}{\partial x} \mathrm{d}x\mathrm{d}s = 0\,,\ \sum M_P = \int_L \frac{\partial(\tau_{x\xi} t)}{\partial x}\,\mathrm{d}x\,\mathrm{d}s + \frac{\mathrm{d}M_t}{\mathrm{d}x}\mathrm{d}x + m_P \mathrm{d}x = 0\,; \quad (16)$$

where $m_P = m_P(x)$ are the moments of torsion per unit length with respect to the pole P. Referring to Eqs.(11) and (13)

$$E_x I_z \frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + E_x I_{z\omega}\frac{\mathrm{d}^4\alpha}{\mathrm{d}x^4} = 0\,, E_x I_{\omega z}\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} + E_x I_\omega \frac{\mathrm{d}^4\alpha}{\mathrm{d}x^4} = m_\omega\,;\ m_\omega = m_P + \frac{\mathrm{d}M_t}{\mathrm{d}x}\,;$$

$$I_z = \int_A y^2 \mathrm{d}A,\quad I_\omega = \int_A \omega^2 \mathrm{d}A,\quad I_{z\omega} = I_{y\omega} = \int_A y\omega \mathrm{d}A, \quad (17)$$

Due to symmetry $I_{z\omega} = I_{y\omega} = 0$. Thus

$$\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = 0\,,\quad E_x I_\omega \frac{\mathrm{d}^4\alpha}{\mathrm{d}x^4} = m_\omega\,. \quad (18)$$

5. Internal forces and stresses

Integration of the shear stresses over the cross-sections gives

$$\int_A \tau_{x\xi}\cos\varphi\,\mathrm{d}A = 0\,,\qquad M_\omega = \int_A \tau_{x\xi} h_P\,\mathrm{d}A \quad (19)$$

where $M_\omega = M_\omega(x)$ is the warping moment. Substitution of Eq. (14) gives

$$\frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = 0\,,\ M_\omega = -E_x I_\omega \frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3}\,;\ \cos\varphi\,\mathrm{d}A = \cos\varphi\, t\mathrm{d}s = t\mathrm{d}y\,,\ \int_L S_z^* \mathrm{d}\omega = I_{z\omega} = 0\,,\ \int_L S_\omega^* \mathrm{d}\omega = I_\omega \quad (20)$$

Referring to Eqs. (18)

$$\mathrm{d}M_\omega / \mathrm{d}x = -m_\omega\,. \quad (21)$$

Thus, by substituting Eq.(14) into Eq. (20)

$$\tau_{x\xi} = \frac{M_\omega S_\omega^*}{I_\omega t}\,. \quad (22)$$

Integration of the normal stresses over the cross-sections gives

$$\int_A \sigma_x\, y\,\mathrm{d}A = 0\,,\qquad B = \int_A \sigma_x \omega \mathrm{d}A, \quad (23)$$

where $B = B(x)$ is the bimoment. By substituting Eqs.(11) and (22)

$$E_x I_z \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - M_z^\omega = 0\,,\quad B = -E_x I_\omega \frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} - B^\omega\,, \quad (24)$$

where

$$M_z^{\omega} = -m_{\omega}\frac{E_x}{GI_{\omega}}\int_L \frac{S_z^* S_{\omega}^*}{t}\mathrm{d}s\,, \qquad B^{\omega} = m_{\omega}\frac{E_x}{GI_{\omega}}\int_L \left(\frac{S_{\omega}^*}{t}\right)^2 \mathrm{d}A\,, \tag{25}$$

Due to symmetry $\int_L \frac{S_z^* S_{\omega}^*}{t}\mathrm{d}s = 0$. Thus, $M_z^{\omega} = 0$. Referring to Eqs. (18) and (20)

$$E_x I_z \frac{\mathrm{d}^3 v}{\mathrm{d}x^3} = \frac{\mathrm{d}M_{\omega}}{\mathrm{d}x} = 0\,, \quad -E_x I_{\omega}\frac{\mathrm{d}^3\alpha}{\mathrm{d}x^3} = \frac{\mathrm{d}B}{\mathrm{d}x} + \frac{\mathrm{d}B^{\omega}}{\mathrm{d}x} = M_{\omega}\,, \tag{26}$$

$$-E_x I_{\omega}\frac{\mathrm{d}^4\alpha}{\mathrm{d}x^4} = \frac{\mathrm{d}^2 B}{\mathrm{d}x^2} = \frac{\mathrm{d}M_{\omega}}{\mathrm{d}x} = -m_{\omega}\,. \tag{27}$$

The normal stress given by Eq. (11), according to Eqs. (22) and (25), finally can be expressed

$$\sigma_x = \frac{B}{I_{\omega}}\omega + \frac{B^{\omega}}{I_{\omega}}\omega - \frac{E_x}{G}\cdot\frac{m_{\omega}}{I_{\omega}}\int_0^s \frac{S_{\omega}^*}{t}\mathrm{d}s\,. \tag{28}$$

The component B^{ω} given by Eq. (25) can also be written as

$$B^{\omega} = \frac{E_x I_{\omega}}{GI_P}\kappa_{\omega} m_{\omega}\,, \quad \kappa_{\omega} = \frac{I_P}{I_{\omega}^2}\int_A \left(\frac{S_{\omega}^*}{t}\right)^2 \mathrm{d}A\,, \quad I_P = \int_A h_P^2\,\mathrm{d}A\,, \tag{29}$$

where κ_{ω} is the shear factor with respect to the α-displacements; I_P is the polar second moment of area. Then, the normal stress can be expressed as

$$\sigma_x = \frac{B}{I_{\omega}}\omega + \frac{E_x\kappa_{\omega}}{GI_P}m_{\omega}\omega - \frac{E_x}{GI_{\omega}}m_{\omega}\int_0^s \frac{S_{\omega}^*}{t}\mathrm{d}s\,. \tag{30}$$

6. Differential equations with separated displacements

Eqs. (24), according to Eqs. (29), can be expressed as

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = 0\,, \quad \frac{\mathrm{d}^2\alpha}{\mathrm{d}x^2} = -\frac{B}{E_x I_{\omega}} - \frac{\kappa_{\omega}}{GI_P}m_{\omega}\,. \tag{31}$$

According to Eqs. (4)

$$\frac{\mathrm{d}^2\alpha_t}{\mathrm{d}x^2} = -\frac{B}{EI_{\omega}}\,, \quad \frac{\mathrm{d}^2\alpha_a}{\mathrm{d}x^2} = -\frac{\kappa_{\omega}}{GI_P}m_{\omega}\,. \tag{32}$$

Integrating, taking into account Eqs. (5) and (21)

$$\frac{\mathrm{d}\alpha_a}{\mathrm{d}x} = -\theta_a = \frac{M_{\omega}\kappa_{\omega}}{GI_P}\,. \tag{33}$$

Integration constants are ignored. It is assumed that the angular displacements θ_a and γ_a do not depend on the boundary conditions. Then

$$E_x I_{\omega}\frac{\mathrm{d}^3\alpha_t}{\mathrm{d}x^3} = -\frac{\mathrm{d}B}{\mathrm{d}x} = -M_{\omega}\,, \quad E_x I_{\omega}\frac{\mathrm{d}^4\alpha_t}{\mathrm{d}x^4} = -\frac{\mathrm{d}B}{\mathrm{d}x} = -\frac{\mathrm{d}M_{\omega}}{\mathrm{d}x} = m_{\omega}\,, \tag{34}$$

Integrating the Eqs. (33)

$$\alpha_a = \frac{\kappa_\omega}{GI_P} B + C . \tag{35}$$

where C is integration constant.

7. Boundary conditions

For starting section A:

$$\alpha_a = 0, \qquad C = -\frac{\kappa_\omega B_A}{GI_P}, \quad \alpha = \alpha_t + \frac{\kappa_\omega (B - B_A)}{GI_P}, \tag{36}$$

where B_A is the bimoment at $x = x_A$. For simply supported beams

$$\alpha\big|_{x=x_{A,B}} = \alpha_t\big|_{x=x_A,B} = 0 , \quad \mathrm{d}^2\alpha_t/\mathrm{d}x^2\big|_{x=x_{A,B}} = 0 \quad \left(B_{A,B} = 0\right); \tag{37}$$

For the clamped beams

$$\alpha\big|_{x=x_A} = \alpha_t\big|_{x=x_A} = 0 . \quad \mathrm{d}\alpha_t/\mathrm{d}x_{x=x_A} = 0;$$

$$\alpha\big|_{x=x_B} = \alpha_t\big|_{x=x_B} + \frac{\kappa_\omega EI_\omega}{GI_P}\left(-\mathrm{d}^2\alpha_t/\mathrm{d}x^2\big|_{x=x_B} + \mathrm{d}^2\alpha_t/\mathrm{d}x^2\big|_{x=x_A}\right) = 0, \qquad \mathrm{d}\alpha_t/\mathrm{d}x\big|_{x=x_B} = 0 . \tag{38}$$

8. Shear factors

The normal stresses, according to Eq. (30), and the displacements, according to Eq. (36), can be expressed as

$$\sigma_x = \lambda \frac{B}{I_\omega}\omega, \qquad \alpha = \eta\alpha_t . \tag{39}$$

where for beams loaded by uniformly distributed moments per unit length for the beam midspan

$$\eta = 1 + \frac{I_t\kappa_\omega}{I_P}\cdot\frac{2(\cosh v - 1)}{v^2\cosh v - 2\cosh v + 2}, \quad \lambda = 1 + \frac{I_t\kappa_\omega}{I_P}\cdot\frac{1}{\cosh v - 1}\left(1 - \frac{I_P b^2}{12 I_\omega \kappa_\omega}\right), \tag{40}$$

for simply supported beams;

$$\eta = 1 + \frac{I_t\kappa_\omega}{I_P}\cdot\frac{2(\cosh v - 1)}{v\sinh v - 2\cosh v + 2}, \quad \lambda = 1 + \frac{I_t\kappa_\omega}{I_P}\cdot\frac{v}{\sinh v - v}\left(1 - \frac{I_P b^2}{12 I_\omega \kappa_\omega}\right), \quad v = \frac{l}{2}\sqrt{\frac{GI_t}{E_x I_\omega}}, \tag{41}$$

for clamped beams. For simple double symmetrical cross-sections (Fig.2)

$$\kappa_\omega = \frac{6}{5}, \; I_\omega = \frac{1}{24}A_1 h^4\rho^2, \; I_P = \frac{1}{2}A_1 h^2, \; I_t = \frac{1}{3}A_1 t_1^2\left(2 + \psi^3\rho^2\right) \tag{42}$$

where $A_1 = bt_1$, $A_0 = ht_0$, $\psi = A_0/A_1$, $\rho = b/h$.

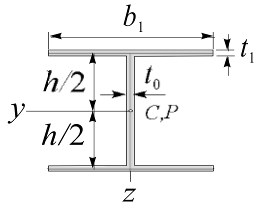

Figure 2. Simple double symmetrical cross-sections

9. Illustrative examples

The factors η and λ, obtained analitically using Eqs. (38) and (40), for simply supported and clamped beams are compared with numerically obtained results by applying the finite element method using ADINA software. The results at the beam midspan are presented in Tab. 2 and Tab. 3. The 9-noded shell elements are used for the FEM analysis. The cross-section properties are defined according to Fig. 2: $b = h = 100$ mm, $t_0 = t_1 = 5$ mm. The distributed line load of 1 kN/m is applied to act as in-plane load at flanges of the cross-section. The material models are analysed with following properties: for orthotropic material ($E_x = 53.78$ GPa, $G = 8.96$ GPa) and for isotropic material ($E = 210$ GPa, $\nu = 0.3$). The analyses are performed for two different beam lengths ($l/b = 3$ and $l/b = 5$).

l/b	Orthotropic				Isotropic			
	Simply supported		Clamped		Simply supported		Clamped	
	(38)	FEM	(38)	FEM	(38)	FEM	(38)	FEM
3	1.089	1.086	1.266	1.238	1.038	1.038	1.115	1.086
5	1.032	1.029	1.096	1.082	1.014	1.013	1.041	1.026

Table 2. The factors λ, according to (38), for orthotropic and isotropic material, respectively

l/b	Orthotropic				Isotropic			
	Simply supported		Clamped		Simply supported		Clamped	
	(40)	FEM	(40)	FEM	(40)	FEM	(40)	FEM
3	1.640	1.655	4.202	4.110	1.277	1.292	2.388	2.354
5	1.230	1.232	2.154	2.116	1.100	1.104	1.500	1.478

Table 3. The factors η, according to (40), for orthotropic and isotropic material, respectively

10. Conclusion

An analytical solution for torsion of thin-walled beams under the influence of shear for double symmetrical cross-sections is given. The shear factors are given in the parametric form in order to compare the shear influence on the beam torsion both for orthotropic and isotropic materials. The shear influence in the case of unidirectional orthotropic beams is significant, and must be taken into account, even in the case of higher beam aspect ratios. Several examples are analysed in comparison with the finite element method. Excellent agreements of the results are obtained.

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