STRENGTH OPTIMISATION OF ORTHOTROPIC PLATES THROUGH INVARIANT FORMULATION OF PHENOMENOLOGICAL FAILURE CRITERIA

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Abstract

Several studies on strength optimisation of anisotropic materials have been done using phenomenological failure criteria that describe the failure with a unique condition that algebraically is presented by a quadratic form, like for strain energy. This work represents an analytical approach to the optimization of orthotropic materials considering the Tsai-Hill, Hoffman and Tsai-Wu criteria. The first step concerns the formulation of these criteria through invariant formulation using the polar method, while the second step concerns the optimization of strength with respect to the orthotropic material orientation. Results obtained using the three different criteria are compared in order to evaluate the three formulations with respect to strength behaviour.

1 Introduction

Many works have been devoted to maximise the stiffness of thin composite structures, see [1, 2, 3, 4]. A more complicate problem concerns the maximisation of strength.

Several studies on strength optimisation of anisotropic materials have been done using the phenomenological failure criteria of Tsai-Hill [5], Hoffman [6] and Tsai-Wu [7]. This is due to the formulation of these criteria based upon a unique condition that algebraically can be interpreted to as a quadratic form, like for strain energy. Normally, studies on stiffness optimisation use an energetic formulation to construct an objective function that usually is a global quantity. Strength optimisation, on the other side, presents a basic difference: any objective function, constructed using a failure criterion, describes a local quantity. Therefore, a large number of studies on strength optimisation of laminates present a local approach to the problem that in addition concerns a number *n* of objective functions, one for each ply. For example, Groenwold and Haftka [8] developed an analytical approach to the optimal strength design of laminates, where Tsai-Wu and Tsai-Hill failure indexes were used. Hence, a number *n* of objective functions, were considered in order to minimise the local value of the maximum failure index. Otherwise, Majak and Hannus [9] formulated the Tsai-Hill and Tsai-Wu failure criteria in terms of strains for orthotropic materials. The failure index was assumed

as the objective function and an analytical method, for 2D orthotropic materials, was proposed to calculate its minimum.
The present work presents an analytical approach to the optimisation of strength for orthotropic materials considering the Tsai-Hill, Hoffman and Tsai-Wu criteria. The first objective is the formulation of these criteria through invariant formulation using the polar method. In particular, the orientation of the material symmetry and the type of orthotropy are explicit terms of the polar parameters, unlike the classical Cartesian representation. The second objective is the optimization of strength (using the Tsai-Hill, Hoffman and Tsai-Wu criteria) with respect to the orthotropic material orientation using the polar method.

2 Classical and invariant formulations

Several failure criteria have been developed for composites materials. Almost all of them are formulated for a typical kind of anisotropic material: the orthotropic one. A brief presentation of the Tsai-Hill, Hoffman and Tsai-Wu criteria is reported, in order to understand the best way to formulate an objective function (that take into account the strength properties of the material). The reference system $\Re$: $\{0;\ x_1,\ x_2,\ x_3\}$ will be considered coincident with the material frame, having x_1 and x_2 lying on the plane of the layer.

1.1 General matrix formulation

We can express the three criteria in the bi-dimensional space, by the general condition:

$$F_{...} = \{\sigma\}^T [F] \{\sigma\} + \{\sigma\}^T \{f\} \leq 1, \tag{1}$$

$$\text{with } [F] = \begin{bmatrix} F_{xx} & F_{xy} & 0 \\ F_{xy} & F_{yy} & 0 \\ 0 & 0 & F_{ss} \end{bmatrix},\ \{\mathrm{f}\} = \begin{Bmatrix} f_x \\ f_y \\ 0 \end{Bmatrix}. \tag{2}$$

A peculiarity of the terms that compose the matrix [*F*] concerns the position of strength properties at the denominator in each fraction. So, we can assert that [*F*] describes, in some sense, the inverse of the strength of the material: the *weakness*. Hence, we can consider the matrix [*F*] as the analogous, for what concerns strength, of the compliance matrix [*S*]; then we will call [*F*] the weakness matrix. We will express [*F*] and, for consistency, the vector {*f*} in a frame rotated by an angle $\pi/2$ around x_3 with respect to $\Re$. In this way we will have the matrix [*F*] described in the new reference system $\Re'$: $\{0;\ x,y,z = x_3\}$ with the *x* axis coincident with the direction of maximum weakness.
The values of matrix and vectorial terms for each criterion are reported in Tables 1 and 2.

	Tsai-Hill	Hoffman	Tsai-Wu
F_{xx}	$\frac{1}{Y^2}$	$\frac{1}{Y_t Y_c}$	$\frac{1}{Y_t Y_c}$
F_{xy}	$-\frac{1}{2X^2}$	$-\frac{1}{2X_t X_c}$	$\frac{F_{12}^*}{\sqrt{X_t X_c Y_t Y_c}}$
F_{yy}	$\frac{1}{X^2}$	$\frac{1}{X_t X_c}$	$\frac{1}{X_t X_c}$
F_{ss}	$\frac{1}{S^2}$	$\frac{1}{S^2}$	$\frac{1}{S^2}$

Table 1. Terms of [*F*] for the three criteria.

	Tsai-Hill	**Hoffman**	**Tsai-Wu**
f_x	0	$\dfrac{Y_c - Y_t}{Y_t Y_c}$	$\dfrac{Y_c - Y_t}{Y_t Y_c}$
f_y	0	$\dfrac{X_c - X_t}{X_t X_c}$	$\dfrac{X_c - X_t}{X_t X_c}$
f_s	0	0	0

Table 2. Terms of {*f*} for the three criteria.

The terms in the tables represent:
- *X*, longitudinal strength along the fibres orientation;
- *Y*, transverse strength perpendicular to the fibres orientation;
- *S*, pure shear strength.

Moreover, subscripts *t* and *c* stand for tension and compression respectively.

2.2 Tensorial formulation

In order to use the polar formalism, we will pass, in the rest of the work, to the tensorial representation of the weakness matrix [*F*] and of the weakness vector {*f*} for an orthotropic material with respect to the reference system $\Re'$: *{x, y, z}*. The correspondence between the Voigt's and the tensor components are:

$$\begin{cases} F_{xx} = F_{xxxx} \\ F_{xs} = 2F_{xxxy} \\ F_{xy} = F_{xxyy} \\ F_{ss} = 4F_{xyxy} \\ F_{ys} = 2F_{yyxy} \\ F_{yy} = F_{yyyy} \end{cases} \quad \text{and} \quad \begin{cases} f_x = f_{xx} \\ f_y = f_{yy} \\ f_s = 2 f_{xy} \end{cases} \tag{3}$$

2.3 About polar representation in ordinary orthotropy

The polar method was introduced by Verchery [10] in 1979. His original work was focused on the representation of a fourth order tensor, like the elasticity tensor, through polar invariants. There are several advantages in using the polar method. First of all the possibility to use invariants with a physical meaning (they have a direct link with the elastic symmetries) and secondly the possibility to change reference system in an easier way than with the Cartesian representation, see [11].

As tacitly assumed in classical criteria, like Tsai-Hill, Hoffman and Tsai-Wu, we also consider that the components of [*F*] and {*f*} correspond respectively to the components of a fourth rank tensor ***F*** and of a second order tensor ***f***, the first one possessing all the tensorial symmetries of a classical elasticity tensor and the second one being symmetric.

The Cartesian components of the orthotropic tensor ***F*** in the plane *(x, y)* can be expressed by the polar ones, see [10], $\gamma_0, \gamma_1, (-1)^l \lambda_0, \lambda_1, \omega_1$:

$$\begin{cases} F_{xxxx} = \gamma_0 + 2\gamma_1 + (-1)^l \lambda_0 \cos(4\omega_1) + 4\lambda_1 \cos(2\omega_1) \\ F_{xxxy} = (-1)^l \lambda_0 \sin(4\omega_1) + 2\lambda_1 \sin(2\omega_1) \\ F_{xxyy} = -\gamma_0 + 2\gamma_1 + (-1)^l \lambda_0 \cos(4\omega_1) \\ F_{xyxy} = \gamma_0 - (-1)^l \lambda_0 \cos(4\omega_1) \\ F_{yyxy} = -(-1)^l \lambda_0 \sin(4\omega_1) + 2\lambda_1 \sin(2\omega_1) \\ F_{yyyy} = \gamma_0 + 2\gamma_1 + (-1)^l \lambda_0 \cos(4\omega_1) - 4\lambda_1 \cos(2\omega_1) \end{cases} \tag{4}$$

The polar parameters $\gamma_0, \gamma_1, \lambda_0, \lambda_1$ are invariants. In particular γ_0, γ_1 represent the isotropic part of the tensor $\boldsymbol{F}$, while λ_0, λ_1 represent the amplitude of the anisotropic part, so λ_0, λ_1 are modules and can't be negatives, see [10]. The polar angle ω_1 represents the orthotropy orientation and l determines the shape of orthotropy. There are two types of orthotropy that depends on the value of l: *0* or any other even value or *1* and any other odd value.
Let us also consider the second order tensor $\boldsymbol{f}$. The Cartesian components can be expressed using the polar ones, γ, λ, ω:

$$\begin{cases} f_{xx} = \gamma + \lambda \cos(2\omega) \\ f_{yy} = \gamma - \lambda \cos(2\omega) \\ f_{xy} = \lambda \sin(2\omega) \end{cases} \tag{5}$$

The polar parameters γ and λ are invariants, while the polar angle ω gives the orientation, with respect to the reference system, of the principal components of $\boldsymbol{f}$ and is hence, frame dependent. In addition, similarly to what happens for the fourth order tensor, γ represents the spherical part of f while λ the deviatoric one.
In the following paragraphs we will introduce also the polar parameters of the stress tensor σ, that will be denoted by T, R, Φ, obviously linked to the Cartesian components of σ in the way described by eq. (5).

3 Optimal material orientation to maximize the strength

In this section we consider the bi-dimensional anisotropy optimisation in order to maximise the strength with respect to the orthotropy direction ω_1. The objective function is the failure criterion of Tsai-Hill or Hoffman/Tsai-Wu for a given stress field. The aim is to find the orthotropic orientation that minimises the objective function written by the aid of the polar formalism. As mentioned, the choice of this formalism is due to the possibility of describing the material symmetries in a very direct way, thanks to the polar invariants, but also to the advantage in having the material orientation ω_1 as an explicit term in the expression of the objective function.

3.1 Optimisation of Tsai-Hill failure index

The Tsai-Hill criterion in the case of ordinary orthotropy is written using the polar parameters of tensor $\boldsymbol{F}$ and of tensor σ:

$$F_{Hill} = 4R^2\gamma_0 + 8T^2\gamma_1 + 4(-1)^l \lambda_0 R^2 \cos 4(\omega_1 - \Phi) + 16TR\lambda_1 \cos 2(\omega_1 - \Phi) \tag{6}$$

where the relation between the polar parameters T, R and the principal stress components σ_I, σ_{II} is:

$$
\begin{aligned}
T &= \frac{\sigma_I + \sigma_{II}}{2} \\
R &= \frac{\sigma_I - \sigma_{II}}{2}
\end{aligned}
\tag{7}
$$

The polar angle Φ represents the direction of the higher principal stress component. The optimization problem can be defined as follow:

$$
\min_{\omega_1} F_{Hill}(\gamma_0, \gamma_1, \lambda_0, \lambda_1, l, \omega_1, T, R, \Phi) \tag{8}
$$

Problem (8) is completely analogous to the minimisation problem of the compliance W_c, see [3], the only design variable being the angle ω_1. The next step concerns the analytical search of the stationary points of F_{Hill} through the direction of the maximum weakness direction, represented by ω_1. So, we have to write:

$$
\frac{\partial F_{Hill}}{\partial \omega_1} = 0 \tag{9}
$$

Eq. (9) is satisfied for one of the following conditions:

$$
\begin{cases}
R = 0 : \text{spherical stress field,} \\
\lambda_0 = \lambda_1 = 0 : \text{isotropic material,} \\
\sin 2(\omega_1 - \Phi) = 0 \Rightarrow \omega_1 - \Phi = \left\{0, \frac{\pi}{2}\right\} \\
\cos 2(\omega_1 - \Phi) = -\frac{\lambda_1 T}{(-1)^l \lambda_0 R}, \text{ with } \frac{|T|}{R} \le \frac{\lambda_0}{\lambda_1}
\end{cases}
\tag{10}
$$

The first two cases are trivial and exclude any possible optimisation of the strength by varying the orthotropy direction, but the last two give three different local minima to be compared:

$$
\begin{aligned}
x_a &= \omega_1 \text{ such that } \omega_1 = \Phi \\
x_b &= \omega_1 \text{ such that } \omega_1 = \Phi + \pi / 2 \\
x_c &= \omega_1 \text{ such that } \omega_1 = \Phi \pm \frac{1}{2} \arccos\left[-(-1)^l \frac{\lambda_1 T}{\lambda_0 R} \right]
\end{aligned}
\tag{11}
$$

For sake of brevity in Fig. 1, we show a summary of the solutions, for more details see [12]. In the figure, α and β represent

$$
\alpha = dir\left(\min\left\{|\sigma_I|, |\sigma_{II}|\right\}\right) \text{ and } \beta = \frac{1}{2} \arccos\left[-(-1)^l \frac{\lambda_1 T}{\lambda_0 R} \right] \tag{12}
$$

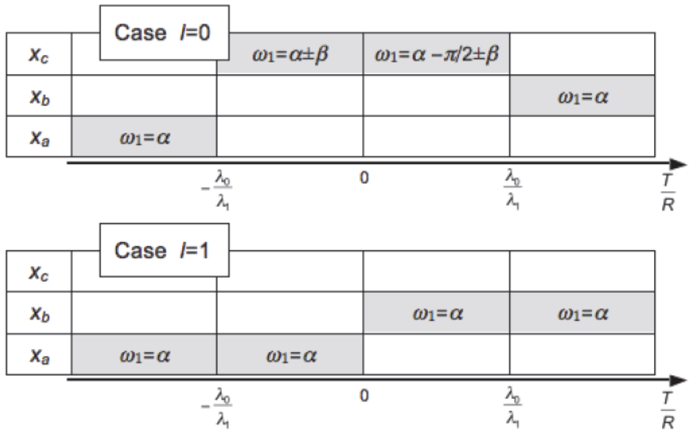

Figure 1. Optimal orthotropy orientation of the weakness tensor for the case of the Tsai-Hill criterion.

The use of the angle α is useful in strength problems, as it indicates the direction of least stress for the material. This figure shows that while for $|T|/R > \lambda_0/\lambda_1$ the optimal solution ω_1^{opt} is equal for the two types of orthotropy (*l = 0* and *l = 1*), $|T|/R < \lambda_0/\lambda_1$ the solution ω_1^{opt} for *l* = 0 becomes anti-optimal for *l* = 1. Therefore, the type of orthotropy plays a decisive role in the optimization of strength.

The polar formulation of the Tsai-Hill failure criterion, very close to that of the compliance, leads the strength optimisation to the same type of solutions obtained for the stiffness optimisation. Particularly, for *l = 0* one optimal orientation of ω_1 depends upon α, while the other one depends also upon the stress tensor (*T*, *R*) and the anisotropic part of the weakness tensor ***F*** (λ_0, λ_1). In this sense, the solution is qualitatively equal to that of maximal stiffness but the actual values of the orthotropy direction minimising compliance and strength may be different, in general. On the other side, for *l=1* the solutions give an orientation of ω_1 aligned with the principal stress component that has the minimum absolute value. Hence, in this case the optimal orientation of an orthotropic material to maximise the strength is equal to the one maximising the stiffness.

A remark to end this section: the above results show clearly, the link between the anisotropy strength properties of the material and the stress field for obtaining the optimal orientation of the material. A similar result is found when the Hoffmann and Tsai-Wu criteria are considered, see for instance [12].

4 Conclusions

The present work concerns an invariant analytical approach to the optimisation of strength in orthotropic materials.

The polar formulation of the three criteria has been considered as the objective function, while the ordinary weakness orthotropy direction has been considered as the optimization variable. We have shown the possibility of having two different groups of solutions, depending upon the type of orthotropy. For the two groups, we derive analytically the different solution with respect to the polar components of the failure criteria. Results show that the type of orthotropy plays a decisive role in the optimization of strength and that, depending on the values of the stresses and of the polar parameters of the failure criteria, the optimal orientation of the material that maximises strength can be equal or different to the one that maximise stiffness and can also be the same or different for the three considered criteria. This means that it is possible to obtain in some cases an orthotropic plate that is simultaneously optimised with respect to two important engineering requirements, stiffness and strength.

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