AVERAGING OF THERMOELASTICITY EQUATIONS OF SOME TYPES OF HETEROGENEOUS MEDIA

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Keywords: inhomogeneity, thermoelasicity, relaxation time

Abstract
The paper considers the stochastically inhomogeneous thermoelastic medium in which each component has of its elastic and thermal parameters and is described by Duhamel-Neumann law. The task of the study is to find a system of equations for the averaged strain and temperature and also to evaluate the effect of fluctuations of the thermoelastic field on the behavior of the medium. The averaged equations of thermo elasticity and thermal balance are constructed in the correlation approximation, i.e. when the physical characteristics of the phases differ little. It is shown that in Duhamel-Neuman law appears the term, dependent on time. An isothermal and adiabatic effective elastic moduli and the relaxation time are defined and also is represented method of evaluation of the stress fluctuations in stochastically inhomogeneous thermoelastic media.

1 Introduction

Composite materials are those materials which consist of components with different physical properties, i.e. it is a heterogeneous medium. Usage of components with different properties makes it possible to produce materials with necessary physical parameters. However, for successful producing such materials it is required a mathematical model that would allow one to predict these properties. The theory of composite materials may be based on different principles, but in order to solve a wide range of problems it must be based on the theory of random functions. If the size of inhomogeneities that make up the composite material is much less than the size of the material, it is possible to use the ergodic hypothesis according to which the value averaged over ensemble equals the value averaged over volume. The paper sets the task to find a common system of equations for the stress, strain and temperature averaged over the ensemble. For such medium the material parameters, i.e. the modulus of elasticity, coefficient of thermal stresses, thermal conductivity and heat capacity are random functions of coordinates. Therefore field parameters such as stress, strain and temperature will also be random functions. The common system of equations of equilibrium and heat balance is given by:

$$\boldsymbol{L}(\boldsymbol{r})\boldsymbol{u}(\boldsymbol{r})=-f(\boldsymbol{r}) \quad (1)$$

or expanded

$$\begin{bmatrix} L_{ik} & L_{i4} \\ L_{4k} & L_{44} \end{bmatrix} \begin{bmatrix} u_k \\ \theta \end{bmatrix} = - \begin{bmatrix} f_i \\ w \end{bmatrix}$$

$$L_{ik} = \frac{\partial}{\partial x_i} \lambda(\boldsymbol{r}) \frac{\partial}{\partial x_k} + 2\,\delta_{ik} \frac{\partial}{\partial x_l} \mu(\boldsymbol{r}) \frac{\partial}{\partial x_l}; \qquad L_{i4} = -\frac{\partial}{\partial x_i} \beta(\boldsymbol{r});$$

$$L_{4k} = p\, T_0 \beta(\boldsymbol{r}) \frac{\partial}{\partial x_k}; \qquad L_{44} = \frac{\partial}{\partial x_i} \kappa(\boldsymbol{r}) \frac{\partial}{\partial x_i} + p\, c_\epsilon(\boldsymbol{r}). \tag{2}$$

Here $\boldsymbol{u}_i(\boldsymbol{r})$ are a components of the displacement, θ is the temperature, $f_i(\boldsymbol{r})$ are components of mass forces, w is the intensity distributed heat source, $\lambda(\boldsymbol{r})$, $\mu(\boldsymbol{r})$ are the Lame coefficients, $\rho(\boldsymbol{r})$ is the density, $\beta(\boldsymbol{r})$ is thermal coefficient of stress $\kappa(\boldsymbol{r})$ is the thermal conductivity, $c_\varepsilon(\boldsymbol{r})$ is the heat capacity at constant strain, δ_{ik} is the Kronekker`s symbol, p is the parameter is the Laplace transform. For indexes that are repeated summation from 1 to 3 is made, angular brackets denotes statistical averaging of physical parameters.

2 Strain relaxation in the statistically inhomogeneous thermo-elastic medium.

Let us make averaging over the ensemble in equation (2)

$$\langle \boldsymbol{L}(\boldsymbol{r}) \boldsymbol{u}(\boldsymbol{r}) \rangle = -f(\boldsymbol{r}), \tag{3}$$

and present all the random function as a sum of mathematical expectations and fluctuations,

$$\boldsymbol{L}(\boldsymbol{r}) = \langle \boldsymbol{L} \rangle + \boldsymbol{L}'(\boldsymbol{r}); \;\; \boldsymbol{u}(\boldsymbol{r}) = \langle \boldsymbol{u}(\boldsymbol{r}) \rangle + \boldsymbol{u}'(\boldsymbol{r}) \tag{4}$$

then we obtain

$$\langle \boldsymbol{L} \rangle \langle \boldsymbol{u} \rangle + \langle \boldsymbol{L}' \boldsymbol{u}' \rangle = -f(\boldsymbol{r}) \tag{5}$$

Subtracting from equation (1) equation (5) we have

$$\boldsymbol{L}' \langle \boldsymbol{u} \rangle + \langle \boldsymbol{L} \rangle \boldsymbol{u}' + (\boldsymbol{L}' \boldsymbol{u}' - \langle \boldsymbol{L}' \boldsymbol{u}' \rangle) = 0. \tag{6}$$

In correlation approximation, ie, when the physical parameters of different components differ not much we can take

$$\boldsymbol{L}' \boldsymbol{u}' - \langle \boldsymbol{L}' \boldsymbol{u}' \rangle \approx 0\,, \tag{7}$$

then equation (6) will become

$$\langle \boldsymbol{L} \rangle \boldsymbol{u}' = -\boldsymbol{L}' \langle \boldsymbol{u} \rangle. \tag{8}$$

It can be considered as a equations of equilibrium for a homogeneous medium with averaged parameters. The solution of equation (8) can be represented as

$$\boldsymbol{u}'(\boldsymbol{r}) = \boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r_1}) * \boldsymbol{L}'(\boldsymbol{r_1}) \langle \boldsymbol{u}(\boldsymbol{r_1}) \rangle \tag{9}$$

$\boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r_1})$ is the Green's tensor of equation of equilibrium for infinite thermoelastic medium, $*$ is the convolution operator, ie $\boldsymbol{G} * \boldsymbol{L} = \int \boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r_1}) \boldsymbol{L}(\boldsymbol{r_1}) dV_1$. All of equation (1) – (9)

contains two parts: deformation and temperature, i.e., $\langle \boldsymbol{u} \rangle = [\langle u_k \rangle, \langle \theta \rangle]^T$, a Green' stensor can be represented as follows

$$\boldsymbol{G}(\boldsymbol{r}) = \begin{bmatrix} G_{ik}(\boldsymbol{r}) & G_{i4}(\boldsymbol{r}) \\ G_{4k}(\boldsymbol{r}) & G_{44}(\boldsymbol{r}) \end{bmatrix} \tag{10}$$

Let us write the system of equations (9) in expanded form

$$u_i' = G_{ik} * \left(\frac{\partial}{\partial x_k} \lambda' \langle \varepsilon_{nn} \rangle + \frac{\partial}{\partial x_l} 2\mu' \langle \varepsilon_{kl} \rangle - \frac{\partial}{\partial x_k} K' \langle \theta \rangle \right) - G_{i4} * p \left(T_0 \beta' \langle \varepsilon_{kk} \rangle - c_\varepsilon' \langle \theta \rangle \right), \tag{11}$$

$$\theta' = G_{4k} * \left(\frac{\partial}{\partial x_k} \lambda' \langle \varepsilon_{nn} \rangle + \frac{\partial}{\partial x_l} 2\mu' \langle \varepsilon_{kl} \rangle - \frac{\partial}{\partial x_k} K' \langle \theta \rangle \right) - G_{44} * p \left(T_0 \beta' \langle \varepsilon_{kk} \rangle - c_\varepsilon' \langle \theta \rangle \right). \tag{12}$$

We can replace the displacement with deformation in equation (11) and use the known property of the integral operator $\boldsymbol{G}$ for infinite medium $\boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r}_1)\nabla_1 = \nabla \boldsymbol{G}(\boldsymbol{r} - \boldsymbol{r}_1)$ [1], then we obtain

$$\varepsilon_{ij}' = G_{ijkl} * \left(\lambda' \langle \varepsilon_{nn} \rangle \delta_{kl} + 2\mu' \langle \varepsilon_{kl} \rangle - \beta' \langle \theta \rangle \right) - G_{ij4} * p \left(T_0 \beta' \langle \varepsilon_{kk} \rangle - c_\varepsilon' \langle \theta \rangle \right), \tag{13}$$

The behavior of a medium has been investigated under static stress, without external heating. Therefore we can assume that $\langle \theta \rangle = 0$. However, $\theta' \neq 0$ due to adiabatic heating, which will vary in different components of heterogeneous medium. Therefore, from (11) and (12) we obtain

$$\varepsilon_{ij}' = G_{ijkl} * \left(\lambda' \langle \varepsilon_{nn} \rangle \delta_{kl} + 2\mu' \langle \varepsilon_{kl} \rangle \right) - G_{ij4} * p T_0 \beta' \langle \varepsilon_{kk} \rangle \tag{14}$$

$$\theta' = G_{4kl} * \left(\lambda' \langle \varepsilon_{nn} \rangle \delta_{kl} + 2\mu' \langle \varepsilon_{kl} \rangle \right) - G_{44} * p T_0 \beta' \langle \varepsilon_{kk} \rangle \tag{15}$$

where

$$G_{ijkl} = () = \frac{1}{2} \left[\frac{\partial^2 G_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 G_{jk}}{\partial x_i \partial x_l} \right], \ G_{ij4} = \frac{1}{2} \left[\frac{\partial G_{i4}}{\partial x_j} + \frac{\partial G_{j4}}{\partial x_i} \right], \ G_{4kl} = \frac{1}{2} \left[\frac{\partial G_{4k}}{\partial x_l} + \frac{\partial G_{4l}}{\partial x_k} \right] \tag{16}$$

It will be convenient to allocate separately bulk and deviator part of the tensor ε_{ij}' in equation (13):

$$\varepsilon_{kk}' = G_{kkll}(\boldsymbol{r} - \boldsymbol{r}_1, p) * 3\mathrm{K}'(\boldsymbol{r}_1) \langle \varepsilon_{nn}(p) \rangle - G_{kk4}(\boldsymbol{r} - \boldsymbol{r}_1, p) * p T_0 \beta'(\boldsymbol{r}_1) \langle \varepsilon_{nn}(p) \rangle \tag{17}$$

$$\varepsilon_{12}'^{d} = G_{1212}^{d}(\boldsymbol{r} - \boldsymbol{r}_1, p) * 2\mu'(\boldsymbol{r}_1) \langle \varepsilon_{12}^{d}(p) \rangle \tag{18}$$

In order to find the dependence between stresses and strains let us multiply equation (17) by β', equation (14) by K',and equation (18) by μ' and average over ensemble, then we will have

$$\langle K' \varepsilon' \rangle = 3 G_{kkll}(\boldsymbol{r} - \boldsymbol{r}_1, p) * \langle K'(\boldsymbol{r}) K'(\boldsymbol{r}_1) \rangle \langle \varepsilon_{nn} \rangle - p T_0 G_{kk4}(\boldsymbol{r} - \boldsymbol{r}_1, p) * \langle K'(\boldsymbol{r}) \beta'(\boldsymbol{r}_1) \rangle \langle \varepsilon_{nn} \rangle \tag{19}$$

$$\langle \mu' \varepsilon_{ij}'^{d} \rangle = 2 G_{ijkl}^{d}(\boldsymbol{r} - \boldsymbol{r}_1, p) * \langle \mu'(\boldsymbol{r}) \mu'(\boldsymbol{r}_1) \rangle \langle \varepsilon_{ij}'^{d} \rangle \tag{20}$$

$$\langle \beta' \theta' \rangle = G_{4kk}(\boldsymbol{r} - \boldsymbol{r}_1, p) * \langle \beta'(\boldsymbol{r}) K'(\boldsymbol{r}_1) \rangle \langle \varepsilon_{nn} \rangle - G_{44}(\boldsymbol{r} - \boldsymbol{r}_1, p) * p T_0 \langle \beta'(\boldsymbol{r}) \beta'(\boldsymbol{r}_1) \rangle \langle \varepsilon_{nn} \rangle \tag{21}$$

Equations (19), (20) and (21) consist two-point moments of random functions $K(\boldsymbol{r}), \mu(\boldsymbol{r}), \beta(\boldsymbol{r})$

$$\begin{aligned}&\langle K'(\boldsymbol{r})K'(\boldsymbol{r_1})\rangle=\langle K'^2\rangle\varphi(|\boldsymbol{r}-\boldsymbol{r_1}|)\quad\langle\mu'(\boldsymbol{r})\mu'(\boldsymbol{r_1})\rangle=\langle\mu'^2\rangle\varphi(|\boldsymbol{r}-\boldsymbol{r_1}|);\\&\langle K'(\boldsymbol{r})\beta'(\boldsymbol{r_1})\rangle=\langle K'\beta'\rangle\varphi(|\boldsymbol{r}-\boldsymbol{r_1}|)\quad\langle\beta'(\boldsymbol{r})\beta'(\boldsymbol{r_1})\rangle=\langle\beta'^2\rangle\varphi(|\boldsymbol{r}-\boldsymbol{r_1}|),\end{aligned}\tag{22}$$

and [1]

$$\phi(r)=\exp\left(-\frac{r}{a_m}\right).\tag{23}$$

Here a_m is the average size of micro-heterogeneity. Equation (19) - (21) should be integrated over the variable $\boldsymbol{r_1}$ across space. However, integration is more convenient to do in Fourier space. Let us make in these equations Fourier transform, we have

$$\begin{gathered}\langle K'\varepsilon'_{kk}\rangle=M_K\langle\varepsilon_{kk}\rangle,\ \langle\mu'\varepsilon'^d\rangle=M_\mu\langle\varepsilon^d\rangle,\ \langle\beta'\varepsilon'_{kk}\rangle=M_\beta\langle\varepsilon_{kk}\rangle;\\M_K=\int[G_{kkll}(\boldsymbol{k},p)\langle K'^2\rangle-pT_0G_{kk4}(\boldsymbol{k},p)\langle K'\beta'\rangle]\phi(k)dV_k;\\M_\mu=\int G_{1212}(\boldsymbol{k},p)\langle\mu'^2\rangle\phi(k)dV_k;\\M_\beta=\int[G_{4kk}(\boldsymbol{k},p)\langle K'\beta'\rangle-pT_0G_{44}(\boldsymbol{k},p)\langle\beta'^2\rangle]\phi(k)dV_k\end{gathered}\tag{24}$$

Fourier transformants of the Green's tensor components are

$$\begin{gathered}G=\begin{bmatrix}G_{ik}&G_{i4}\\G_{4k}&G_{44}\end{bmatrix},\ G_{ik}(\boldsymbol{k})=\frac{1}{\langle\mu\rangle k^2}\left(\delta_{ik}-\frac{\Lambda+\langle\mu\rangle}{\Lambda+2\langle\mu\rangle}\frac{k_ik_k}{k^2}\right),\ G_{i4}(\boldsymbol{k})=\frac{ik_i(\Lambda-\langle\lambda\rangle)}{pT_0\beta k^2(\Lambda+2\langle\mu\rangle)},\\G_{4k}(\boldsymbol{k})=\frac{ik_k(\Lambda-\langle\lambda\rangle)}{\beta k^2(\Lambda+2\langle\mu\rangle)},\ G_{44}(\boldsymbol{k})=\frac{(\Lambda-\langle\lambda\rangle)(\langle\lambda+\mu\rangle)}{pT_0\beta^2(\Lambda+2\langle\mu\rangle)},\ \Lambda=\langle\lambda\rangle+\frac{pT_0\langle\beta\rangle^2}{\langle\kappa\rangle k^2+p\langle c_\varepsilon\rangle}.\end{gathered}\tag{25}$$

In respect that with the Fourier transform $\frac{\partial}{x_i}f(\boldsymbol{r})\to-ik_if(\boldsymbol{k})$ let us find the Fourier transform for G_{kkll}, G_{kk4}, G_{4kk}, G_{44} and φ:

$$\begin{gathered}G_{kkll}=\frac{\langle c_\varepsilon\rangle}{\langle c_\varepsilon\rangle(\langle\lambda+2\mu\rangle)+T_0\langle\beta\rangle^2}+\frac{a}{p+b};\ G_{kk4}=\frac{T_0\langle\beta\rangle}{c_\varepsilon(\langle\lambda+2\mu\rangle)+T_0\langle\beta\rangle^2}-\frac{\tilde{a}}{p+b};\\G_{4kk}=pT_0G_{kk4};\quad G_{44}=G_{kk4}(\langle\lambda+\mu\rangle)/\langle\beta\rangle\\a=\frac{T_0\langle\beta\rangle^2\langle\kappa\rangle k^2}{[\langle c_\varepsilon\rangle(\langle\lambda+2\mu\rangle)+T_0\langle\beta\rangle^2]^2};\ b=\frac{(\langle\lambda+2\mu\rangle)\langle\kappa\rangle k^2}{c_\varepsilon(\langle\lambda+2\mu\rangle)+T_0\langle\beta\rangle^2};\ \tilde{a}=\frac{(\langle\lambda+2\mu\rangle)T_0\langle\beta\rangle\langle\kappa\rangle k^2}{[\langle c_\varepsilon\rangle(\langle\lambda+2\mu\rangle)+T_0\langle\beta\rangle^2]^2}\\\varphi(\boldsymbol{k})=\frac{a_m^3}{\pi^2(1+k^2a_m^2)^2}.\end{gathered}\tag{26}$$

In the end we need to inverse Laplace transform. As a result, we obtain Hooke's law as follows

$$\sigma_{kk}=3K_S^*\varepsilon_{kk}+\frac{\langle K'^2\rangle a-\langle K'\beta'\rangle\tilde{a}}{15\langle\mu\rangle}\int_0^\infty F(t-t_1)\langle\varepsilon_{kk}(t_1)\rangle dt_1$$
$$\sigma^d=2\mu_S^*\varepsilon^d+\frac{\langle\mu'^2\rangle}{15\langle\mu\rangle}a\int_0^t F(t-t_1)\langle\varepsilon^d(t_1)\rangle dt_1 \quad (27)$$
$$F(t)=\int_0^\infty\frac{x^4e^{-btx^2}}{(1+x^2)^2}dx$$

Stress depends on time due to heat, which appears due to adiabatic heating during deformation, $\tau=b^{-1}$ is the relaxation time. Adiabatic modulus of elasticity will be

$$K_S^*=\langle K\rangle+\frac{1}{4\langle\mu\rangle}\left[\frac{\langle K'^2\rangle\langle\mu\rangle\langle c_\varepsilon\rangle+\langle K'\beta'\rangle T_0\langle\beta\rangle}{c_\varepsilon\langle\lambda+2\mu\rangle+T_0\langle\beta\rangle^2}\right],$$
$$\mu_S^*=\langle\mu\rangle+\frac{\langle\mu'^2\rangle}{15\langle\mu\rangle}\left[6+\frac{4\langle\mu\rangle\langle c_\varepsilon\rangle}{\langle c_\varepsilon\rangle\langle\lambda+2\mu\rangle+T_0\langle\beta\rangle^2}\right]. \quad (28)$$

Isothermal modules can be obtained from (28) at β=0.

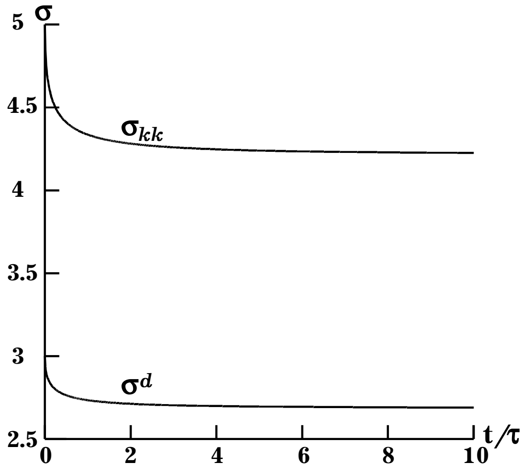

Figure 1. Time Dependence of stress for heterogeneous medium at a constant strain.
σ_{kk} bulk stress σ_d shear.

For example, Figure 1 shows the time dependence of stress under constant strain. As you can see, the process of relaxation of stress is not only for bulk stress but also for the shear. This is due to the fact that even in the absence of macroscopic compression, microscopic compression in components is still present. As shown in figure above the quick deformation (shock) and slow deformation for the same solid will be different.

3 Fluctuations of stresses in a heterogeneous medium

We consider the problem of thermoelastic state of an inhomogeneous medium under homogeneous heating without external stress. As is well known for homogeneous media pressure in this case will be equal to zero, and deformation will be caused by thermal expansion. For the isothermal case, ie when $p=0$ we have

$$\langle\sigma_{kk}\rangle=0,\ \theta'=0,\ \theta\neq0,\ \langle\varepsilon_{kk}\rangle=\frac{\langle\beta\rangle}{K_T^*}\theta. \quad (29)$$

But the strain fluctuations are not equal to zero due to the difference between the coefficients of thermal stresses in components. In order to estimate the fluctuations of the stress let us distinguish bulk part of the stress in equation (13) and use (29), then we obtain

$$\sigma'_{kk}=K'\varepsilon'_{kk}=G_{kkll}(\boldsymbol{r}-\boldsymbol{r_1})*K'(\boldsymbol{r})M'(\boldsymbol{r_1})\theta\,,$$
$$M(\boldsymbol{r_1})=\frac{K(\boldsymbol{r_1})}{K'_T}\langle\beta\rangle-\beta(\boldsymbol{r}). \tag{30}$$

Now let us find the mean square fluctuation of bulk stress

$$\langle\sigma^2_{kk}\rangle=G_{kkll}(\boldsymbol{r}-\boldsymbol{r_1})*G_{kkll}(\boldsymbol{r}-\boldsymbol{r_2})\langle K'^2(\boldsymbol{r})M'(\boldsymbol{r_1})M'(\boldsymbol{r_2})\rangle\theta^2 \tag{31}$$

In equation (31) the convolution is made of two variables $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ After integration over these variables we receive.

$$\langle\sigma'^2\rangle=M_\sigma\theta^2\,,$$
$$M_\sigma=\frac{18\langle K'^2M'^2\rangle}{\langle 3\mathrm{K}+4\mu\rangle}. \tag{32}$$

Formula (32) represents the fluctuations of stresses in an inhomogeneous medium under uniform heating [2].

Conclusions.

Heterogeneous media, such as composite materials cannot always be considered as homogeneous with some effective material parameters. Heterogeneous media have new properties, in particular the relaxation term appears the law of Hooke. The relaxation time depends on the size of the micro-structure. Moreover, the uniform heating in heterogeneous mediums appears the internal tensions caused by different coefficients of thermal expansion of components.

References

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