AN ANALYTICAL CALCULATION METHOD FOR STRESS CONCENTRATIONS IN NOTCHED MULTILAYERED GF/PP-COMPOSITES AND ITS VALIDATION

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Abstract
As a further development of prior research, an analytical method for the determination of the stress-strain fields in the vicinity of holes in multilayered composites has been developed which takes into consideration the influences of a finite outer boundary of the specimen. The method is based on the classical laminate theory and the use of complex-valued potential functions. To account for the shape of the specimen, the method of conformal mappings is applied for the inner boundary, whilst a combination of boundary collocation and least squares method is used for the outer boundary. For the verification of the implementation of the calculation model, experimental and numerical studies have been carried out on textile-reinforced GF/PP plates. Finally, a first approach to implement the method as an analytical sub-model in combination with a global finite-element structural model is presented.

1 Introduction
Structures made from multilayered fibre- or textile-reinforced composites have been gaining in importance over the past years, whereat textile semi-finished reinforcing products are getting more and more into the focus of application. However, Lightweight structures made of multilayered composites are often weakened by necessary cut-outs for example at various force induction points or feedthroughs. Therefore the problems arising from stress concentrations have come more and more into the focus of research, since such local stresses can often be regarded as design drivers for the whole composite structure. To fully utilize the large lightweight design potential of this group of materials, particularly in the case of future-oriented multi-material design methods, the provision of adapted dimensioning concepts for critical areas like cut-outs and notches is indispensable. The use of numerically based FE-modelling techniques for calculating the stresses, strains and displacements in the vicinity of notches is rather unsatisfying since on the one hand the results very much depend on the chosen mesh and time-consuming convergence studies have to be carried out, on the other hand the notched areas are often very small in comparison to the whole structure but the effort to calculate adequate FE-results in these areas is disproportionally great.

To overcome these shortcomings, an adapted calculation concept for notched areas in multilayered composites is provided in this paper that offers the chance to be used as an analytical sub-model in combination with classical finite-element modelling of the overall structure. Therefore, the analytical model not only takes into consideration the hole itself but as well the finite outer boundary of the (sub-)structure.

The model is based on the assumptions of linear elasticity, small deformation gradients, a plane stress state and the classical laminate theory (CLT). Starting with these fundamental assumptions and based on the fundamental works of LEKHNITSKII [1] the method of analytical functions in combination with conformal mappings is used to express the stress, displacement and strain fields in the finite plate with circular or elliptical cut-out by four complex-valued potential functions. The coefficients of the principal and the regular part of the LAURENT series representations of the potential functions are determined by LAURENT series development of the boundary condition on the inner boundary, analytical continuation of the potential functions onto the whole area of the notched plate and using a combination of boundary collocation method and least squares method for fulfilling the boundary conditions on the outer boundary (see e.g. [2-4]). More in-depth information about the theoretical background and the used solution methods can also be found in e.g. [5-8].

The solution method and its implementation have been verified in a number of experimental and numerical studies. Based on the good correlation of the analytical, experimental and numerical results, the method has been used for the combination of a global FE analysis of a notched structure using a coarse regular FE mesh and the developed analytical model as a sub-model. First results are presented in this paper.

2 Analytical calculation methods

2.1 Fundamental equations

In this paper, the mathematical model of the notched multilayered plate with finite outer boundaries is assumed. Beyond that, it is assumed that the specimen is behaving linear-elastically with small deformation gradients and that the applied loads induce a plane stress state. In this case, the well known CLT is used to describe the structural behavior of the multilayered composite, taking into consideration the extension-bending coupling effect occurring in the case of un-symmetric composites

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & & B_{22} & B_{26} \\ & & A_{66} & \mathit{symm.} & & B_{66} \\ \hline & & & D_{11} & D_{12} & D_{16} \\ & \text{symm.} & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{array}\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{1}$$

with

N_i, M_i:	force resultants and moment resultants,
(A_{ij}), (B_{ij}), (D_{ij}):	extensional, extension-bending coupling and bending stiffnesses of multilayered composites,
ε_i^0, γ_j^0, κ_k:	distortions of the neutral plane.

To deal with all possible load-cases in a uniform way, the equilibrium of force and moment resultants known from the classic plate theory by Kirchhoff is supplemented by the membrane force resultants as known from the membrane problem, and a generalized plate equation is derived (see e. g. [5])

$$\underline{\underline{\Delta}}^T \left[\begin{array}{c|c} (A_{ij}) & (B_{ij}) \\ \hline (B_{ij}) & (D_{ij}) \end{array} \right] \underline{\underline{\Delta}} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \underline{\underline{\Delta}}^T \left[\begin{array}{c} (N_i) \\ \hline (M_i) \end{array} \right] = -\underline{P}, \qquad \underline{\underline{\Delta}} = \left[\begin{array}{ccc} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ \partial/\partial y & \partial/\partial x & 0 \\ \hline 0 & 0 & -\partial^2/\partial x^2 \\ 0 & 0 & -\partial^2/\partial y^2 \\ 0 & 0 & -2\partial^2/\partial x\partial y \end{array} \right] \tag{2}$$

and

u_0, v_0, w_0:	displacements of the neutral plane,
$\underline{P} = (0, 0, p)^T$:	vector of external loads.

For dealing with the system of coupled linear partial differential equations (PDES) (2) – which is composed of two partial differential equations of second order and one partial differential equation of fourth order in u_0, v_0 and w_0 – the method of complex-valued potential functions is called upon. This method can be regarded as extension of the method of complex-valued stress functions, which is well established in the theory of plane elasticity [9-11] for nearly a century. Here, the PDES (2) is equivalently transformed into one partial differential equation of order eight in w_0. For the solution of this partial differential equation, the following ansatz is chosen:

$$w_0 = 2\,\mathrm{Re}\left(\sum_{k=1}^{4} \Phi_k\left(\mathfrak{z}_k\right) \right) \tag{3}$$

with four complex potential functions Φ_k referring to the four different complex planes $\mathfrak{z}_k = \mathfrak{z} + \lambda_k \overline{\mathfrak{z}}$. The complex parameters λ_k can be obtained as roots of the characteristic equation of the transformed PDES (2). Since it can be shown that of the eight different complex roots of this characteristic equation always two by pair have to be conjugated complex for real materials, only four roots have to be taken into account [1,7]. With the ansatz (3) the other displacement functions are given by

$$\begin{gathered} u_0 = 2\,\mathrm{Re}\left(\sum_{k=1}^{4} p_k \varphi_k\left(\mathfrak{z}_k\right) \right), \quad v_0 = 2\,\mathrm{Re}\left(\sum_{k=1}^{4} q_k \varphi_k\left(\mathfrak{z}_k\right) \right) \\ \text{with} \quad \varphi_k\left(\mathfrak{z}_k\right) = \left(1+\lambda_k\right) \frac{d\Phi_k\left(\mathfrak{z}_k\right)}{d\mathfrak{z}_k} \quad (k = 1\ldots 4) \end{gathered} \tag{4}$$

with eight parameters $p_k, q_k \in \mathbb{C}$. By using the analytical functions as given in (3) and (4), the displacement functions automatically fulfill the homogeneous PDES (2). So in the next step, the analytical functions only have to be adapted to the boundary conditions of the respective problem.

2.2 Conformal mappings, boundary conditions and solution method

To adapt the analytical displacement functions Φ_k to the given boundary conditions in the case of different circular or elliptical cut-outs, the method of conformal mapping is called upon. Here, the exterior of the given problem in the $\mathfrak{z}$-plane is projected conformally onto the exterior of a unit circle in the ζ-plane. This opens the possibility of a uniform approach for the determination of the respective displacement functions, independently of the actual notch contour by solving the projected problem in the ζ-plane and back-projecting the solution into the $\mathfrak{z}$-plane. In case of circular or elliptical cut-outs as dealt with in this paper, necessary mappings of the $\mathfrak{z}$-plane or the affinely distorted $\mathfrak{z}_k$-planes onto the ζ-plane or the respective ζ_k-plane, or rather the corresponding reverse mappings, are given by

$$\begin{aligned} \mathfrak{z} &= \omega(\zeta) = \frac{a+b}{2}\cdot\zeta + \frac{a-b}{2}\cdot\frac{1}{\zeta} \\ \mathfrak{z}_k &= \omega_k(\zeta_k) = \left(\frac{1+\lambda_k}{2}a + \frac{1-\lambda_k}{2}b\right)\zeta_k + \left(\frac{1+\lambda_k}{2}a - \frac{1-\lambda_k}{2}b\right)\frac{1}{\zeta_k} \end{aligned} \tag{5}$$

with $\zeta \equiv \zeta_k$ and a, b as the semi-axes of the elliptical notch. More details about the used conformal mappings can be found e. g. in [7] or [8].

In the further development of the calculation model, a combination of constant edge forces per unit length N_x, N_y, N_{xy} and moments per unit length M_x, M_y, M_{xy} on the finite outer boundary of a rectangular plate are taken into account as examples for technically relevant external loads or as section forces and section moments derived from a global structural FE analysis of an un-disturbed structure, respectively (Figure 1). At the edge of the notch, the plate is loaded by a combination of a constant normal force per unit length N_i and a constant bending moment per unit length M_i (Figure 1).

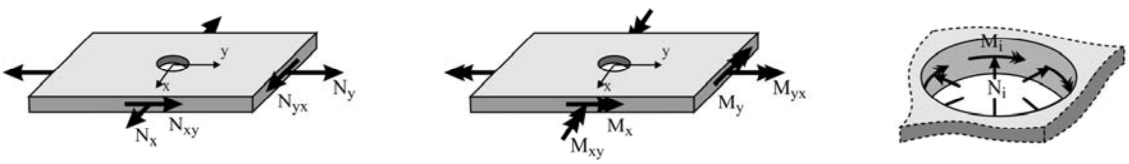


Figure 1: Boundary conditions on the finite outer boundary and on the cut-out

In order to take into consideration the loads on the outer boundary as well as those on the edge of the cut-out, the actual state of stress is decomposed into the three subproblems (Figure 2):

- **I** a finite, unnotched plate with the loads on the outer edge,
- **II** a finite notched plate with loads at the edge of the notch, chosen in such a way that, with superposition of I and II, an overall unloaded notch edge results,
- **III** a finite notched plate with a loaded notch edge according to the given original boundary conditions.

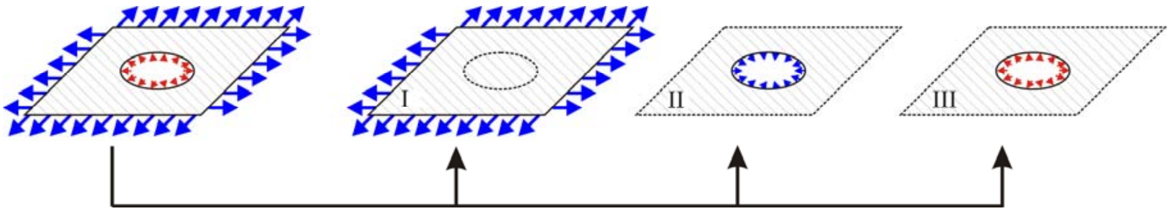


Figure 2: Decomposition of the coupled membrane-bending problem by means of superposition

The solution of subproblem I can be determined elementarily. From this solution, the adapted boundary conditions for subproblem II are taken. For the solution of subproblem II and III, the method of conformal mapping is applied as described earlier in this section. After mapping the problem onto the corresponding ζ_k-planes, the given boundary conditions on the edge of the cut-out as well as the potential functions (3) are expanded into LAURENT series on the edge of the unit circle E

$$\Phi_k\left(\mathfrak{z}_k\right) = \Phi_k\left(\omega_k(\zeta_k)\right) = B_{k0} + \sum_{m=1}^{\infty}\left\{B_{km}\zeta_k^m\right\} + \sum_{m=-\infty}^{-1}\left\{B_{km}\zeta_k^m\right\} \quad (k=1\ldots4). \tag{6}$$

In the case of the finite outer boundary of the specimens here, the regular part of the LAURENT series (6) will not disappear as it does in the case of an infinite plate [8]. So to determine all coefficients of the Laurent series (6), additional equations for the outer finite boundary are needed. These additional equations are generated by analytical continuation of the potential functions Φ_k onto the whole area of the notched plate and using a combination of boundary collocation method and least squares method on the unloaded outer boundary Γ. Therefore, the stress and moment resultants N_i, M_i are expressed with the aid of the LAURENT series representation of the potential functions (6)

$$N_i\left(\mathfrak{z}\right) = N_i^{\Phi}\left(\Phi_k\left(\omega_k(\zeta_k)\right)\right), \quad M_i\left(\mathfrak{z}\right) = M_i^{\Phi}\left(\Phi_k\left(\omega_k(\zeta_k)\right)\right) \quad (k=1\ldots4;\ i=1,2,12), \tag{7}$$

a subset P of discrete points is chosen on the outer boundary and the additional equations for determining all coefficients of the potential functions (6) are taken from the minimization problem

$$\begin{gathered} P = \left\{P_j \in \Gamma\left(j=1,\ldots,n\right)\right\} \\ \sum_{j=1}^{n}\sum_{(i=1,2,12)}\left(N_i^{\Phi}\left(P_j\right)\right)^2 + \sum_{j=1}^{n}\sum_{(i=1,2,12)}\left(M_i^{\Phi}\left(P_j\right)\right)^2 \to \min. \end{gathered} \tag{8}$$

Finally, the analytical calculation method developed here is implemented into a calculation tool and can be used on standard office PCs.

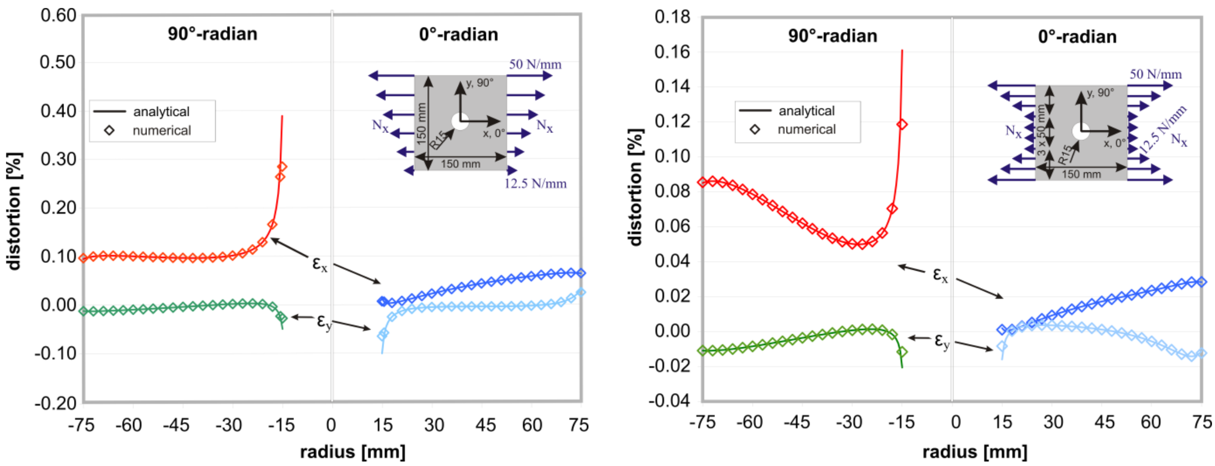
3 Validation of the implementation of the developed calculation method

To verify the presented solution method for stress concentration problems of fibre- and textile-reinforced MLCs considering effects of a finite outer boundary, a number of experimental and numerical finite-element (FE) studies have been carried out and the experimentally or numerically determined strain fields were compared to those calculated by using the developed analytical methods. The material data of the notched plate used in these studies were experimentally determined from multilayered $[0/90]_s$-GF/PP-plates which were produced at the ILK within the scope of Collaborative Research Centre SFB 639. The bidirectionally reinforced single lamina, consisting of knitted fabrics, was made from hybrid glass-fibre-polypropylene yarns and the material properties of the single lamina were as follows:

$$E_1 = 21.5\,GPa,\, E_2 = 20.7\,GPa,\, G_{12} = 1.86\,GPa,\, \nu_{12} = 0.13. \tag{9}$$

Since the experimental investigations, which show a very good correlation of experimentally measured and analytically calculated results, have already been published e. g. in [8], the authors will not repeat the results here.

For the numerical investigations and validation of the implemented calculation method, the finite-element systems ABAQUS and ANSYS have been used. Here, exemplarily the results of one of the conducted studies – on linearly varying stress and moment resultants on the finite outer boundary – are given in Figure 3. In these figures, the analytically calculated and numerically determined decay behavior of the ε_x- and ε_y-strain-curves for different non-constant boundary conditions along 0°- and 90°- radian for a quadratic [0/90]s-GF/PP-plate are shown.



(a) varying boundary load I (b) varying boundary load II

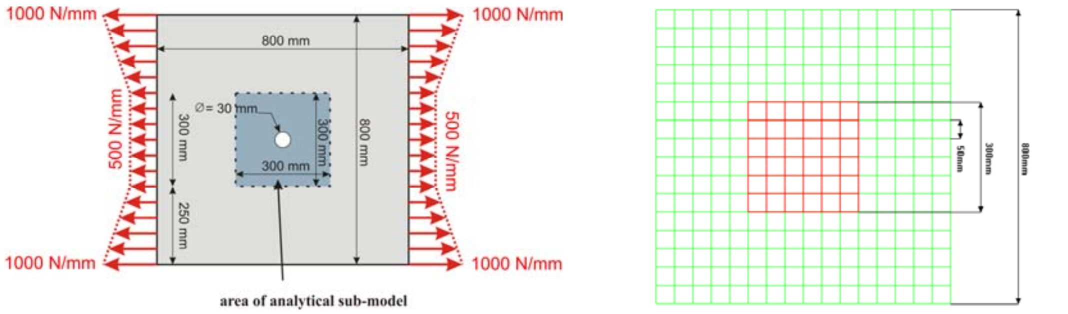
Figure 3: Comparison of analytically calculated and numerically determined strains ε_x, ε_y along 0°- and 90°-radian for different linearly varying boundary conditions on the finite outer boundary of a notched quadratic $[0/90]_s$-GF/PP-plate ($l = w = 150$ mm)

4 Application of the calculation method as a sub-model of a global FE-analysis

The developed calculation method offers great potential to be used as an analytical sub-model in combination with a global FE analysis for notched structures. This approach would help to reduce the lengthy and tedious step of generating problem-adapted meshes – as needed in the FE method – and therefore, it will support dimensioning critical areas in multilayered composites caused by cut-outs.

In the first step of this approach, a FE analysis of the given problem is carried out using a coarse regular FE mesh and ignoring the cut-outs. In the second step, an area around the notches is defined in such a way that the influence of the disturbance has decayed on the boundary of this area. Then, the stress and moment resultants on this boundary are transferred from the “global” FE model to the analytical model as boundary conditions on the finite outer boundary. In the final step, the analytical model is used to calculate the local stress, strain and displacement fields in the neighbourhood of the notch. In this paper, results using this approach are presented for the first time.

As an example, the results for the analysis of a quadratic plate (800 *mm* x 800 *mm*) under a varying boundary load according to Figure 4 (a) are given. For the analytical sub-model, an area of 300 *mm* x 300 *mm* around the cut-out has been chosen (Figure 4 (b)). To verify the approach, the results generated by the analytical sub-model are compared to results from a “classical” FE analysis of the notched structure using a mesh verified by a convergence study. In Figure 5, a comparison of the decay behaviour of ε_x-, ε_y-, γ_{xy}-distortion along the 45°- and the 90°-radian is shown. The analytical results are taken from the analytical sub-model whilst the numerical results are taken from the “classical” FE analysis. A good correlation of the results determined by the new approach of analytical sub-modelling and by classical FE analysis can be observed.



(a) dimension and loads (b) “global” FE-model of undisturbed plate

Figure 4: Quadratic plate under varying boundary load as an example of an analytical sub-model

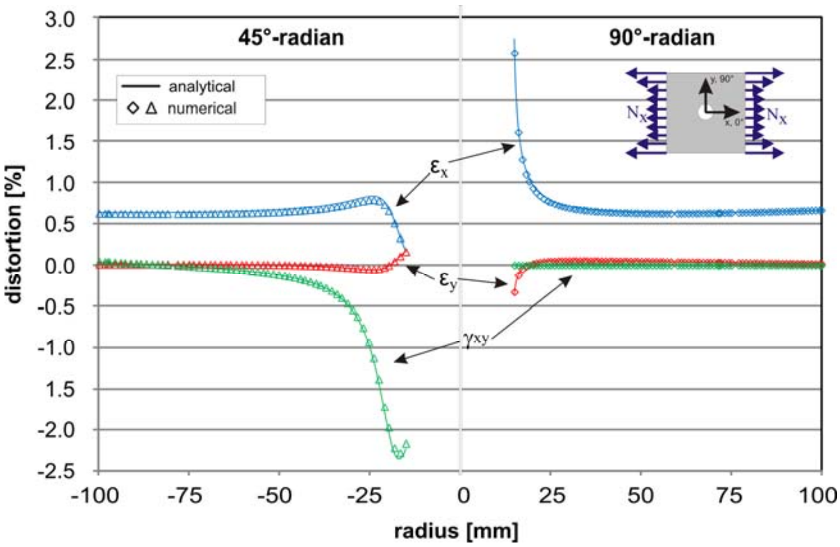


Figure 5: Comparison of the decay behaviour of distortions determined by analytical sub-modelling and classical FE analysis

5 Conclusion

For the problem of multilayered composites with circular or elliptical cut-outs and a finite outer boundary, a solution method for the stress-strain analysis has been developed using complex-valued potential functions, the method of conformal mapping and boundary collocation in combination with the least squares method. With respect to using the developed method as an analytical sub-model in combination with FE techniques, continuous, piecewise linear stress and moment resultants can be applied as applicable loads on the outer boundary.

A number of experimental and numerical analyses were performed for the verification of the developed calculation methods and its implementation. The comparison shows a very good correlation of the experimentally or numerically determined results and the results obtained by means of the developed solution method (see e.g. [8,12]).

Based on the good results, the combination of FE analysis of notched multilayered composites with a coarse regular mesh and the developed calculation method using the stress and moment resultants of the FE analysis of an undisturbed specimen as input for a local analysis of the neighbourhood of the notch has been demonstrated in first studies. This approach will help to reduce the lengthy and tedious analysis of notch-induced stresses and therefore, it will support dimensioning critical areas in multilayered composites caused by cut-outs.

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