HIGH ORDER MULTIPOINT APPROXIMATIONS OF STOCHASTIC ELASTIC BOUNDARY VALUE PROBLEM FOR POLYDISPERSE COMPOSITES

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Abstract
The aim of this work is research of statistical characteristics of structural stress and strain fields in composites with random structure on a base of developing of elasticity theory stochastic boundary value problem solution methods. Stochastic structure of composite is described with multipoint correlation functions, which are obtained for the realizations of synthesized 3D structure models with real materials characteristics specified. Statistical characteristics of structural stress and strain fields in the components are determined from boundary value problem solution, equations and boundary conditions of which contain stochastic variables. New analytical expressions for average values and dispersions of stress and strain fields in components of material with numerical results for simple shear case of macroscopic state of strain are presented.

1 Introduction
One of the urgent research directions in composite mechanics is developing of mathematical models which allow calculation of stress and strain fields’ parameters for each component of composite. This is necessary for prediction of deformation and fracture mechanisms in dependence on geometric parameters of construction, loading conditions and structure of material. The structural model of the material considered in this work represents two-phase composite, which consists of matrix and randomly located spherical inclusions with random radius.

For randomly reinforced composites studying, methods based on random functions theory are used. The advantages of such methods are that they allow taking into account such important factor of real composite structure as randomness of components, relative collocation and statistical spreading of their properties.

Thus, stochastic stress and stress fields’ characteristics in elements of composite can be determined from boundary value problem solution using statistical properties of structure and loading conditions. Statistical information about the composite structure is carried by multipoint correlation functions, which can be obtained by structure modeling or from real composite specimen analysis.

The objective of this work is to develop research methods of stochastic structural stresses and strains fields in composites with random structure on a base of elastic boundary value problem solution with high-order correlation functions.

2 Boundary value problem solution and statistical characteristics of deformation fields

It is assumed that physical and geometrical quantities, which describe composite's properties, are statistical homogenous and ergodic random fields, matrix and inclusions are elastic and homogenous, adhesion on borders between components is ideal, body forces are not taken into account, small strains are considered, media is macroscopically homogenous.
Thus, the boundary value problem of elasticity theory for some representative volume can be defined:

$$\sigma_{ij,j}(\vec{r}) = 0, \tag{1}$$

$$\varepsilon_{ij}(\vec{r}) = \frac{1}{2}\left(u_{i,j}(\vec{r}) + u_{j,i}(\vec{r})\right), \tag{2}$$

$$\sigma_{ij}(\vec{r}) = C_{ijkl}(\vec{r})\,\varepsilon_{kl}(\vec{r}), \tag{3}$$

$$u_i(\vec{r})\Big|_{\vec{r}\in\Gamma_u} = \varepsilon^*_{ij} r_j, \tag{4}$$

where $C_{ijkl}(\vec{r})$ – tensor of structural elasticity modulus, $\varepsilon^*_{ij} = e_{ij}$, e_{ij} – components of constant arbitrarily assigned symmetric tensor of small strains, r_j – coordinates of points on body surface.
Fields of structural elasticity modulus and fields of displacements can be represented as a sum of average component and pulsation:

$$C_{ijkl}(\vec{r}) = \langle C_{ijkl}(\vec{r})\rangle + C'_{ijkl}(\vec{r}), \tag{5}$$

$$U_m(\vec{r}) = \langle U_m(\vec{r})\rangle + U'_m(\vec{r}), \tag{6}$$

where $\langle C_{ijkl}(\vec{r})\rangle$ is the invariable isotropic tensor of structural modulus of elasticity:

$$\langle C_{ijkl}(\vec{r})\rangle = pC^I_{ijkl} + (1-p)C^M_{ijkl}, \tag{7}$$

where p is the inclusions volume concentration, C^I_{ijkl} and C^M_{ijkl} are the elasticity modulus of matrix and inclusions respectively.
One of the solution methods of elasticity theory stochastic boundary value problem is building a system of equations containing correlation functions. The unknown higher-order correlation functions are part of lower-order equations, so the final solution theoretically can be obtained by superposition of some conditions relatively to higher-order correlation functions. With help of the Green function method, as it in detail described in [1, 2], the boundary value problem is reduced to the integral-differential stochastic equation for pulsation of displacements. In the recurrent form it looks as follows:

$$\frac{\partial u_i'^{(\chi)}(\vec{r})}{\partial x_j} = \int_{V_1} \frac{\partial G_{im}(\vec{r},\vec{r}_1)}{\partial x_j} \left[C'_{mnkl}(\vec{r}_1) e_{kl} + C'_{mnkl}(\vec{r}_1) \frac{\partial u_k'^{(\chi-1)}(\vec{r}_1)}{\partial x_l} \right]_{,1n} dV_1 \tag{8}$$

where $G_{im}(\vec{r},\vec{r}_1)$ is the Green function [3, 4], χ is approximation in which the problem is solving.

In the first approximation, pulsations of displacements in the right part of (8) are taken zero. In the second approximation, result obtained from the first approximation is substituted into the right part of the equation. Solutions both in the first and the second approximations are considered. They are used in formulas for statistical characteristics of stress and strain fields, such as average values and dispersions.

In the general view, the formulas for the strain field average values and dispersions in the component C of the composite look as follows:

$$\langle \varepsilon_{ij} \rangle_C = e_{ij} + \frac{1}{\langle \lambda_C \rangle} \langle \lambda'_C(\vec{r}) \varepsilon'_{ij}(\vec{r}) \rangle, \tag{9}$$

$$T^{(\varepsilon)}_{ij\alpha\beta} = \langle \varepsilon'_{ij}(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle_C = \langle \varepsilon'_{ij}(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle + e_{ij} e_{\alpha\beta} - \langle \varepsilon_{ij} \rangle_C \langle \varepsilon_{\alpha\beta} \rangle_C +$$
$$+ \frac{1}{\langle \lambda_C \rangle} \left(\langle \lambda'_C(\vec{r}) \varepsilon'_{ij}(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle + e_{ij} \langle \lambda'_C(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle + e_{\alpha\beta} \langle \lambda'_C(\vec{r}) \varepsilon'_{ij}(\vec{r}) \rangle \right), \tag{10}$$

where $\lambda_C(\vec{r})$ is the indicator function, which is equal 1 if the radius-vector is in the component C and 0 otherwise. Formulas for the stress field $\sigma(\vec{r})$ have the same view.

The mixed moments $\langle \varepsilon'_{ij}(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle$, $\langle \lambda'_C(\vec{r}) \varepsilon'_{ij}(\vec{r}) \rangle$ and $\langle \lambda'(\vec{r}) \varepsilon'_{ij}(\vec{r}) \varepsilon'_{\alpha\beta}(\vec{r}) \rangle$, which are required in (9) and (10), are constructed using boundary value problem solution (8) and multipoint correlation functions.

3 Multipoint correlation functions for composite structure

In the case of matrix composites the indicator function $\lambda(\vec{r})$ is equal 1 if the radius-vector is in one of the inclusions and 0 if it is in the matrix. Random indicator function pulsation $\lambda'(\vec{r})$ can be introduced as:

$$\lambda'(\vec{r}) = \lambda(\vec{r}) - < \lambda(\vec{r}) >. \tag{11}$$

For the boundary value problem solution in the first and the second approximation, the correlation functions up to, respectively, third and fifth order are essential. The n-order correlation function has the following expression through the indicator function pulsation:

$$K^{(n)}_{\lambda}(\vec{r},\vec{r}_1,...,\vec{r}_n) = \langle \lambda'(\vec{r}) \lambda'(\vec{r}_1) ... \lambda'(\vec{r}_n) \rangle. \tag{12}$$

Indicator function $\lambda(\vec{r})$ is statistically homogeneous and isotropic, thus the correlation functions $K^n_{\lambda}(\vec{r},\vec{r}_1,...,\vec{r}_n)$ depend only on a distance between considering points $|\vec{r}-\vec{r}_n|$. Values of correlation functions are obtained for synthesized 3D models of composite structure in dependence of steps $|\vec{r} - \vec{r}_n|$, $|\vec{r}_m - \vec{r}_n|$ with mesh methods: the synthesized stochastic structure fragment is patterned with a mesh, values of indicator function possess 0 or 1 depending on presence of the matrix or one of the inclusions in mesh point.

Moment functions up to fifth order have been built. Further in calculations it is convinient to use normalized correlation functions:

$$f_\lambda^n(\vec{r}, \vec{r}_1, \vec{r}_2, ...) = \frac{K_\lambda^n(\vec{r}, \vec{r}_1, \vec{r}_2, ...)}{D_\lambda^n}, \tag{13}$$

where $D_\lambda^n = (1-p)^n p + (-p)^n (1-p)$ is the central n-order moment.

Fifth-order correlation functions for structures with various inclusions volume concentration are presented on figure 1.

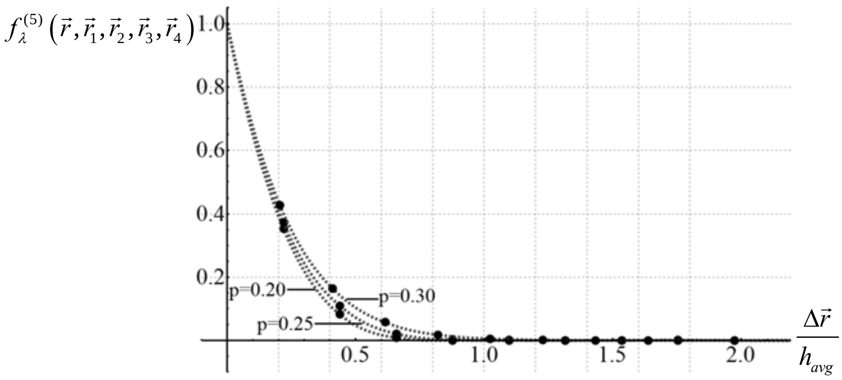

Figure 1. Fifth-order correlation functions for the structures with different inclusions volume concentration p.

The explicit forms of normalized correlation functions are used in combination with boundary value problem solution (8) for statistical characteristics (9) and (10) calculation. To obtain analytical expressions for correlation functions the following approximation expression types were used for second-order (14) and higher-order (15) correlation functions:

$$f_\lambda^{(2)}(\vec{r}, \vec{r}_1) = \exp(c_1, \vec{r}, \vec{r}_1)\left[\cos(c_2, \vec{r}, \vec{r}_1) + c_3 \sin(c_2, \vec{r}, \vec{r}_1)\right], \tag{14}$$

$$f_\lambda^{(n)}(\vec{r}, \vec{r}_1, \vec{r}_2, ..., \vec{r}_n) = \exp(c_1, \vec{r}, \vec{r}_1, \vec{r}_2, ..., \vec{r}_n)\left[\cos(c_2, \vec{r}, \vec{r}_1, \vec{r}_2, ..., \vec{r}_n)\right], \tag{15}$$

where c_i are approximation coefficients, which are determined for correlation function of every order for each realization of 3D structure.

4 Numerical results for deformation fields statistical characteristics in simple shear case

All numerical calculations were implemented in Wolfram Mathematica using parallel computations.

As an example of the method implementation, porous composites with different inclusions volume concentration ($p = 0.15, 0.20, 0.25, 0.30$) are considered with the following elastic properties of matrix: $E_m = 2 \times 10^{11} Pa$, $\nu_m = 0,3$. Components of macroscopic deformation tensor (state of strain) are set as simple shear state of strain case: $e_{12} = e_{21} = 1$ MPa.

Coefficients for correlations functions approximating expressions for the considered structures are presented in table 1.

		Second-order correlation function	Third-order correlation function	Fourth-order correlation function	Fifth-order correlation function
Structure with $p = 0.15$	c_1	51.4459	5.4650	5.5720	5.7638
	c_2	60.8700	7.6007	6.4085	6.6567
	c_3	4.8205	-	-	-
Structure with $p = 0.20$	c_1	52.5380	4.1242	4.7187	4.9281
	c_2	63.0195	3.1281	3.8203	3.9610
	c_3	6.1731	-	-	-
Structure with $p = 0.25$	c_1	52.3713	3.8509	4.2839	4.5152
	c_2	55.3549	2.0138	2.5263	2.6647
	c_3	6.5165	-	-	-
Structure with $p = 0.30$	c_1	57.4423	3.5150	4.0205	4.2141
	c_2	53.8207	1.2450	1.5240	1.5690
	c_3	8.0559	-	-	-

Table 1. Coefficients of correlation functions approximating expressions.

The mixed moments in (9) and (10) consist of multidimensional integrals, which are calculated in Wolfram Mathematica. Table 2 contains values of strain and stress fields' statistical characteristics which are obtained using the boundary value problem solution in the second approximation.

Statistical characteristics	$p = 0.15$	$p = 0.20$	$p = 0.25$	$p = 0.30$
Average strains $< \varepsilon_{12} >_M$, $\times 10^{-6}$	0.98991	0.98507	0.98078	0.97561
Average stresses $< \sigma_{12} >_M$, MPa	0.13254	0.12535	0.11805	0.11070
Dispersions of strains $< \varepsilon'_{12}\varepsilon'_{12} >_M$, $\times 10^{-14}$	1.78269	2.32009	2.90176	3.79986
Dispersions of stresses $< \sigma'_{12}\sigma'_{12} >_M$, MPa^2 $\times 10^{-2}$	0.04742	0.09379	0.15996	0.23979

Table 2. Statistical characteristics of stress and strain fields for porous composites with different inclusions volume concentration in the simple shear case.

These numerical results demonstrate the possibility of deformation fields' statistical characteristics estimation, which is important for structural fracture probability determination and for design reliability analysis.

5 Conclusions

The result of this work is a mathematical model, input data of which are geometry of structure, elastic properties of the components of composite and state of strain. In the output it gives statistical characteristics of strain and stress fields. Geometry of structures is taking into account by means of correlation functions. Material properties can be set by varying elastic constants for matrix and inclusions. The obtained general forms of analytical expressions for the deformation fields' moments in the first and the second approximation allow calculating statistical characteristics in various macroscopic states of strain.

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