SECOND GRADE MODELING FOR THE STRAIN LOCALIZATION ANALYSIS OF LAYERED MATERIALS WITH DAMAGING INTERFACES

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Abstract
A second-order computational homogenization procedure for heterogeneous materials with periodic microstructure is applied to the analysis of a layered strip with damaging interface subjected to simple shear. The second gradient model is applied in a strain localization analysis and localization limiters depending on the geometry and mechanical parameters of the layered material are obtained.

1 Introduction
In multi-scale description of materials with complex microstructure, classical homogenization approaches may have disadvantages and non-local constitutive models may be necessary to include material length scales into the constitutive relations in order to take into account the size of the micro-components and the effects of high stress and strain gradients and to prevent pathological localizations related to the strain-softening constitutive assumption for components or interfaces in the microstructure. Although many non-local constitutive equations are either purely phenomenological or only implicitly incorporate the presence of the underlying microstructure, higher-order computational homogenization techniques seem to be efficient tools to derive micromechanically based non-local constitutive equations [1].
A second-order computational homogenization procedure for heterogeneous materials with periodic microstructure has been recently proposed by the Authors [2,3], that consists of a two steps analysis of the periodic unit cell with properly prescribed boundary conditions. This approach derives from a multi-scale kinematics with a high-continuity representation of the micro-displacement field as superposition of a local macroscopic displacement field, expressed in a polynomial form linearly depending on the macro-strain components, and an unknown micro-fluctuation field accounting for the effects of the heterogeneities. The latter contribution is represented as the superposition of two unknown functions each of which related to the first-order and to the second-order strain, respectively. This kinematical micro-macro framework guarantees that the micro-displacement field is continuous across the interfaces between adjacent unit cells and implies a computationally efficient procedure that applies in two steps.
This methodology is here applied to the analysis of simple shear of a layered strip with boundary constraints. The boundary are orthogonal to the layers, which are assumed to be elastic and with equal thickness but different elastic moduli and interconnected through

elasto-damaging interfaces having linear response in the post-peak phase. A relative displacement normal to the layers is applied to the boundaries so that a one-dimensional problem is obtained. From the second-order homogenization procedure applied to the layered material, the elastic and tangent moduli of the second order continuum are derived. Moreover, through a localization of the stress field at the microscale, the transition from the elastic regime to the inelastic one at the interface is controlled. The solution of the resulting incremental problem is obtained according to the boundary conditions on the displacement components along the layer direction. The solutions and the influence of the constitutive parameters and of the internal length are analyzed and discussed with reference to strain localization effects.

2 Shear strain of a layered body with soft interfaces

Let us consider a layered body obtained as an unbounded d_2-periodic arrangement of two different layers having thickness a and b (here $d_2 = a + b$ is defined), respectively, and length L as shown in Figure 1. The phases are assumed to be isotropic with elastic moduli (E_a, ν_a) and (E_b, ν_b), respectively. The shear response of the thin interface connecting the adjacent layers is represented in terms of the resolved shear stress $\tau = \sigma_{12}$ and the corresponding tangent component $[\![u_1]\!]$ of the displacement jump according to a rigid-softening constitutive a. The interface response is rigid until $\tau \leq \tau^*$, τ^* being the initial limit shear strength of the interface. Once attained this limit value, the damage process takes place in the interface with residual shear strength linearly depending through the softening parameter $h(<0)$ on the displacement jump $[\![u_1]\!]$. The ultimate displacement jump u^* is defined as a further constitutive parameter, that depends on the damage energy $\mathscr{G}^* = \frac{1}{2}\tau^* u^*$. Moreover, in the softening phase an elastic unloading is considered for $[\![\dot{u}_1]\!] < 0$. The obtained heterogeneous model is firstly considered as a Cauchy continuum under the assumption of small strains.

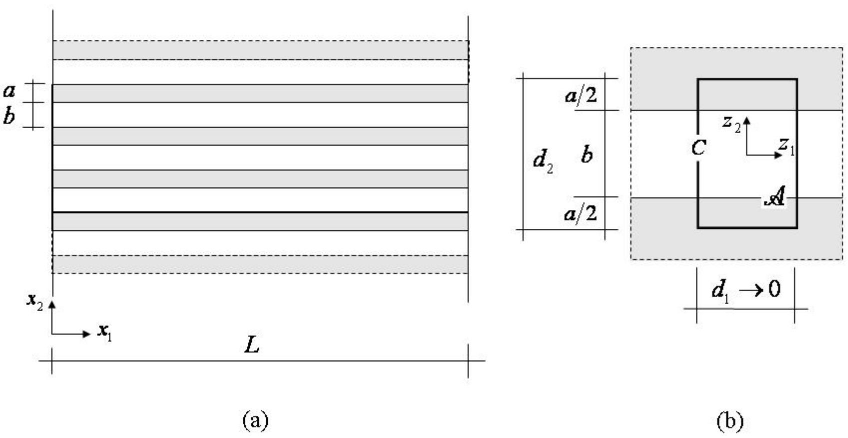

Figure 1. (a) Unbounded periodic layered body; (b) unit cell.

The shear strain in the layered body is analysed with reference to periodic boundary conditions prescribed on the bases $x_1 = 0$ and $x_1 = L$ and neglecting body forces. The overall shear is obtained by prescribing a vanishing average displacement at the left base

$\left\langle u_2\left(0,x_2\right)\right\rangle=0$, being $\left\langle g\left(x_1,x_2\right)\right\rangle=\frac{1}{d_2}\int_0^{d_2} g\left(x_1,x_2\right)dx_2$, and a monotonically increasing average displacement $\left\langle u_2\left(L,x_2\right)\right\rangle=\Delta$ (the ratio Δ/L measuring the overall shear strain in the body). At both the bases a vanishing average displacement is prescribed for the component $\left\langle u_1\right\rangle=0$, with warping allowed, so that an homogeneous stress field in the body is obtained in the first phase of the process when the elastic regimes takes place. Here the d_2-periodic solution is considered, namely $\mathbf{u}\left(x_1,x_2+d_2\right)=\mathbf{u}\left(x_1,x_2\right)$ and $\boldsymbol{\sigma}\left(x_1,x_2+d_2\right)=\boldsymbol{\sigma}\left(x_1,x_2\right)$, where $\mathbf{u}$ and $\boldsymbol{\sigma}$ are the displacement vector and the stress tensor. In the elastic phase the stress field is homogeneous with increasing Δ up to the limit state $\tau=\tau^*$ that is attained simultaneously at each point of the interface. Increasing the prescribed displacement a localization process of the shearing strain takes place, which is the subject of the present analysis.
The solution of this fine-scale problem is computationally expensive so that it is convenient to replace the heterogeneous model here considered with an equivalent homogeneous one to obtain equations whose coefficients are not rapidly oscillating while their solutions are close to those of the original equations. This issue is analysed with reference to the second gradient homogenization which provides constitutive equations equipped with internal lengths as it is shown in the following.

3 Second gradient homogenization of the layered material

Let consider the second gradient homogeneous model that is assumed equivalent to the layered model presented in the previous Section. The macro-displacement field is represented by the displacement components of the periodic cell centered at position $\mathbf{x}$ shown in figure 1.b. From the boundary conditions prescribed on the heterogeneous model it follows that the only non vanishing displacement component is $U_2\left(x_1\right)=\left\langle u_2\right\rangle_{\mathbf{x}}$, which is independent on x_2. The macro-strain field is represented by the non-vanishing component of the displacement gradient tensor $H_{21}=U_{2,1}$, and the strain and rotation tensor $E_{21}=\Omega_{21}=U_{2,1}/2$, respectively, while the non-vanishing component of the second gradient of the displacement is $\kappa_{211}=\partial^2 U_2/\partial x_1^2$ (see for reference Mindlin [4] and Germain [5]).
The internal forces are represented by the corresponding components of the first order symmetric stress tensor $\Sigma_{21}=\Sigma_{12}$ and the components of the second-order stress tensor μ_{211} and μ_{121}. As the material is stratified, the costitutive equations for both the elastic and the incremental inelastic phase are orthotropic with uncoupled response $\Sigma_{21}=2C_{2121}E_{21}$, $\mu_{211}=S_{211211}\kappa_{211}$ and $\mu_{121}=S_{121211}\kappa_{211}$, where C_{2121}, S_{211211} and S_{121211} are the elastic moduli (or inelastic tangent moduli). Both the constitutive moduli and the stress components are independent on x_2 because the d_2-periodicity of the material, while they may depend on x_1 as an effect of the inelastic constitutive response of the interface. Moreover, the non-vanishing components of the real stress tensor are $T_{21}=\Sigma_{21}-\mu_{211,1}=2C_{1212}E_{12}-\left(S_{211211}\kappa_{211}\right)_{,1}$ and $T_{12}=\Sigma_{21}-\mu_{121,1}=2C_{1212}E_{12}-\left(S_{121211}\kappa_{211}\right)_{,1}$ and the equilibrium equation $T_{ij,j}=0$ for the case of vanishing body forces is written in the form $T_{21,1}=\Sigma_{21,1}-\mu_{211,11}=0$. By substitution of the compatibility and constitutive equations, the field equation takes the form

$$C_{1212}U_{2,1}-\left(S_{211211}U_{2,11}\right)_{,1}=T_{21}=const \quad , \tag{1}$$

where the real stress along the specimen is constant.

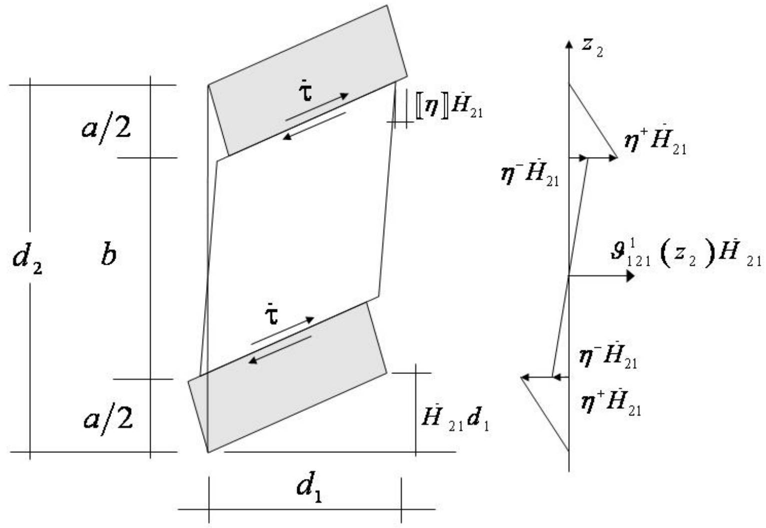

Figure 2. First order shear deformation of the periodic cell with damaged interface (H_{21} and $\kappa_{211}=0$ prescribed).

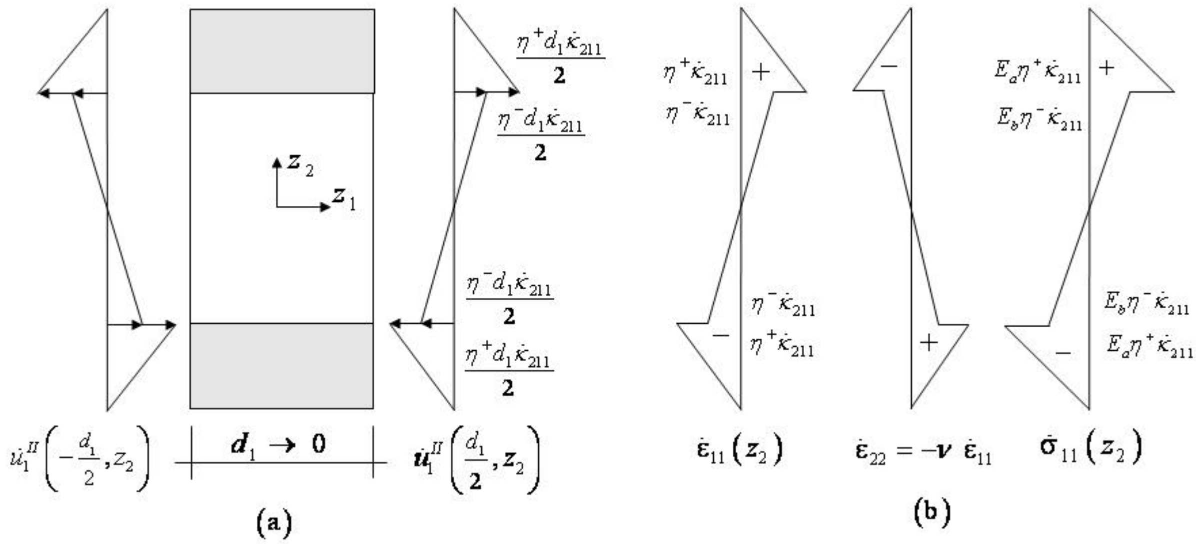

Figure 3(a) Second order prescribed displacements at the boundary C_1 with damaged interface ($H_{21}=0$ and κ_{211} prescribed); (b) Strain and stress field in the cell for $d_1 \to 0$.

To obtain the costitutive moduli of the second grade continuum together with the macro-strain and macro-stress components which are relevant to the study of the layered material, the two-scales kinematics are related through a downscaling of the kinematics from the macroscale to the microscale, according to the methodology proposed in [2]. For the specific problem here considered it may be shown that the downscaling is represented in the form

$$\begin{cases} u_1 = \vartheta^1_{121}\left(H_{21}+\kappa_{211}z_1\right)+\vartheta^2_{1211}\kappa_{211} \\ u_2 = U_2 + H_{21}z_1 + \dfrac{1}{2}\kappa_{211}z_1^2 + \vartheta^2_{2211}\kappa_{211} \end{cases}, \tag{2}$$

where the macro-kinematical descriptors U_2, H_{21}, κ_{211} are prescribed and the functions ϑ^1_{121}, ϑ^2_{1211}, ϑ^2_{2211} represent the micro-fluctuation field related to the macro-strains H_{21} and

κ_{211}, respectively. Once obtained these microfluctuation functions, by an application of the Hill-Mandel macro-homogeneity criterion the constitutive moduli may be obtained together with the stress field. By considering the ratios characterizing the model parameters $\zeta = a/b$, $r = G_a/G_b$ and $\delta = \frac{2G_a}{ha}$, the constitutive moduli and the shear stress acting on the layer interface are given in the following.

Undamaged interface – elastic moduli and interface shear stress

- First order homogenization

$$C_{1212} = \frac{1+\zeta}{\zeta+r}G_a \quad ; \quad \tau = 2\frac{1+\zeta}{\zeta+r}G_a E_{21}.$$

- Second order homogenization

$$S_{211211} = \frac{\mu_{211}}{\kappa_{211}} = \frac{G_a a^2}{6(1+\zeta)r}\left[r\zeta(1+\nu_a)+(1+\nu_b)\right]\left(\frac{r-1}{r+\zeta}\right)^2 \; ; \; \tau = 0.$$

Damaged interface -tangent moduli and interface shear stress rate

- First order homogenization

$$C^t_{1212} = G_a \frac{\zeta+1}{\zeta(\delta+1)+r}; \quad \dot{\tau} = 2G_a \frac{\zeta+1}{\zeta(\zeta+1)+r}\dot{E}_{21}.$$

- Second order homogenization

$$S^t_{211211} = \frac{\dot{\mu}_{211}}{\dot{\kappa}_{211}} = \frac{G_a a^2}{6(1+\zeta)}\left[\zeta(1+\nu_a)\left(\frac{\delta\zeta+r-1}{(\delta+1)\zeta+r}\right)^2 + \frac{1}{r}(1+\nu_b)\left(\frac{1-r+\delta}{(\delta+1)\zeta+r}\right)^2\right] \; ; \; \dot{\tau} = 0.$$

4 Shear strain localization in the layered solid

Let consider now the homogeneous continuum equivalent to the heterogeneous one described in Section 2. The boundary condition applied to the macro-displacement, strain and stresses are deduced from those ones applied at the micro-scale and are given in the following form $U_2(x_1=0)=0$, $\mu_{211}(x_1=0)=0$, $U_2(x_1=L)=\Delta$, $\mu_{211}(x_1=L)=0$ namely fixed displacement at one base and prescribed monotonically increasing displacement Δ with zero second order stress components. In the first phase for $\tau<\tau^*$ the field equation (1) takes the form $U_{2,1111} - \frac{1}{\lambda^2}U_{2,11} = 0$, with $\lambda = \sqrt{\frac{S_{211211}}{C_{1212}}}$ characteristic length of the elastic layered material, and the solution is the classical homogeneous one. The displacement gradient and the first order stress field are homogeneous along the body with vanishing second gradient of the displacement and second order stresses. The real stress equals the tangential stress at the interfaces obtained by down-scaling from the homogeneous model. Accordingly, the limit state $\tau = \tau^*$ is obtained for $H^*_{21} = \tau^*/C_{1212}$ and then for $\Delta^* = H^*_{21}L$ the limit state is attained at each point of the interfaces. At this point the localization of inelastic strain takes place as a consequence of the softening assumption for the interface and may be analysed according the approach proposed by Chambon et al. [6,7].

The incremental response in the localization analysis is obtained by assuming a lateral portion of the strip of length ℓ_e undergoing elastic deformation and a complementary portion of length $\ell_d = L - \ell_e$ undergoing inelastic strain rates. The resulting displacement gradient along the strip results: $H_{21} < H_{21}^*, \forall\ 0 \le x_1 \le \ell_e$ (i.e $\dot{H}_{21} < 0\ \forall\ 0 \le x_1 \le \ell_e$); $H_{21} \ge H_{21}^*, \forall\ \ell_e \le x_1 \le L$ (i.e $\dot{H}_{21} > 0\ \forall\ 0 \le x_1 \le \ell_e$).

The equilibrium equation in the strip are written in incremental forms in terms of the displacement component U_2

$$\begin{aligned} &\dot{U}_{2,1111} - \frac{1}{\lambda^2}\dot{U}_{2,11} = 0, \qquad 0 \le x_1 \le \ell_e \\ &\dot{U}_{2,1111} + \frac{1}{\lambda_d^2}\dot{U}_{2,11} = 0, \qquad \ell_e \le x_1 \le L \end{aligned} \tag{4}$$

where $\lambda_d = \sqrt{-\frac{S_{211211}^t}{C_{1212}^t}}$ is the characteristic length of the damaged layered material. The solution is written in the form

$$\dot{U}_2 = \begin{cases} \dot{U}_2^e = A_e + B_e x_1 + C_e \cosh\left(x_1/\lambda\right) + D_e \sinh\left(x_1/\lambda\right), & 0 \le x_1 \le \ell_e \\ \dot{U}_2^d = A_d + B_d x_1 + C_d \cos\left(x_1/\lambda_d\right) + D_d \sin\left(x_1/\lambda_d\right), & \ell_e \le x_1 \le L \end{cases} \tag{5}$$

where the eight unknown constants are obtained by imposing the four boundary conditions at the strip ends and the continuity conditions at the point separating the elastic and the damaged portion where the shear stress at the interface takes the limit value $\tau = \tau^*$

$$\begin{aligned} &\dot{U}_2^e\left(x_1 = \ell_e\right) = \dot{U}_2^d\left(x_1 = \ell_e\right),\ \dot{H}_{21}^e\left(x_1 = \ell_e\right) = \dot{H}_{21}^d\left(x_1 = \ell_e\right) = 0, \\ &\dot{\mu}_{211}^e\left(x_1 = \ell_e\right) = \dot{\mu}_{211}^d\left(x_1 = \ell_e\right),\ \dot{T}_{21}^e\left(x_1 = \ell_e\right) = \dot{T}_{21}^d\left(x_1 = \ell_e\right). \end{aligned}$$

The extension ℓ_d of the region where inelastic deformation is localized is obtained as the solution of equation

$$\sinh\left(\frac{L-\ell_d}{\lambda}\right)\cos\left(\frac{\ell_d}{\lambda_d}\right)\lambda + \lambda_d \cosh\left(\frac{L-\ell_d}{\lambda}\right)\sin\left(\frac{\ell_d}{\lambda_d}\right) = 0 \tag{6}$$

and is independent on the prescribed displacement Δ. In fact, the assumed linear response of the softening phase makes the overall response of the post-localization of the strip linear with respect to the imposed displacement Δ and the real stress takes the form

$$T_{21}\left(\ell_d, \Delta\right) = C_{1212}\frac{\Delta^*}{L} - \frac{C_{1212}C_{1212}^t\left[\cosh\left(\frac{\ell_e}{\lambda}\right) + \sinh\left(\frac{\ell_e}{\lambda}\right)\right]\cos\left(\frac{\ell_d}{\lambda_d}\right)\cosh\left(\frac{\ell_e}{\lambda}\right)}{\Lambda}\frac{\Delta - \Delta^*}{L} \tag{7}$$

where

$$\begin{aligned} \Lambda = {} & C_{1212}^t\frac{\lambda}{L}\cos\left(\frac{\ell_d}{\lambda_d}\right)\left[\left(1 - \frac{\ell_e}{\lambda}\right)\left(\cosh^2\left(\frac{\ell_e}{\lambda}\right) + \cosh\left(\frac{\ell_e}{\lambda}\right)\sinh\left(\frac{\ell_e}{\lambda}\right)\right) - 1\right] + \\ & + C_{1212}\frac{\lambda_d}{L}\left[\left(\cosh^2\left(\frac{\ell_e}{\lambda}\right) + \cosh\left(\frac{\ell_e}{\lambda}\right)\sinh\left(\frac{\ell_e}{\lambda}\right)\right)\left(\sin\left(\frac{\ell_d}{\lambda_d}\right) - \frac{\ell_d}{\lambda_d}\cos\left(\frac{\ell_d}{\lambda_d}\right)\right)\right]. \end{aligned} \tag{8}$$

From an analysis of the influence of the model parameters on the solution of equation (6) it may be shown that the extension of the strip where the inelastic strain localizes is limited with respect to realist lengths of the specimen. This result seems to substantiate this model in consideration of the criticisms by Jirásek and Rolshoven [8,9] regarding the approach proposed by Chambon et al. [6,7].

5 Numerical example

The layered material is assumed having layers of equal thickness $a = 5$ mm $(\zeta = 1)$ and the strip length is assumed $L/a = 20$. The elastic properties are $G_a = 833.3$ MPa, $r = 2.41$, $\nu_a = 0.2$, $\nu_b = 0.3$, while the interface parameters are $\tau^* = 4.9$ MPa, $h = -1$ N/mm^3, $\delta = -33.3$. The limit displacement gradient is $H^*_{21} = 0.01$. The elastic moduli of the equivalent second order model are $C_{1212} = 489.13$ MPa, $S_{211211} = 5.15$ MPa cm^2, with characteristic length $\lambda = 1.02$ mm, while the tangent moduli obtained in case of damaged interface are $C^t_{1212} = -55.69$ MPa, $S^t_{211211} = 36.34$ MPa cm^2 with characteristic length $\lambda_d = 8.01$ mm.

From the strain localization analysis described in the previous Section the length $\ell_d = 4.87a$ of the damaged portion of the strip is obtained by solving equation (6). In the diagrams of figure 4 the overall model response is represented in terms of non-dimensional real stress T_{21}/C_{1212} versus imposed displacement Δ/L for different values of the strip lengths L/a. These diagrams show a post peak linear response and an increase of the brittleness is observed for increasing the ratio L/a.

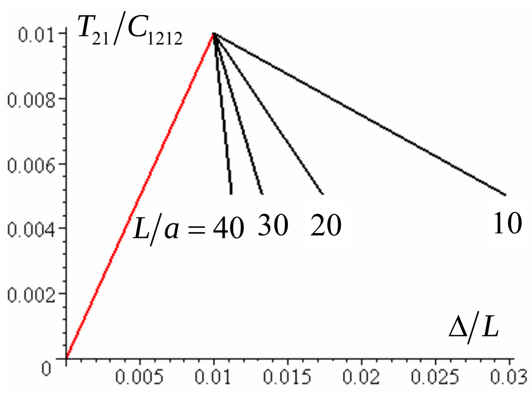

Figure 4. Model response in terms of non-dimensional real stress and prescribed displacement for different values at the L/a.

The strain localization process is shown in the diagrams of figure 5.a where the displacement gradient is represented along the strip for increasing values of the ratio $\Delta/\Delta^* \geq 1$. After reaching the limit state $\Delta/\Delta^* = 1$, in the elastic region (in red) the displacement gradient decreases together with the elastic shear stress in the interfaces; in the damaged region high displacement gradients take place. At the point $x_1 = \ell_e$ a discontinuity in the second gradient of the displacement field is obtained as a consequence of the different second order moduli obtained from the elastic and damaged constitutive model. Finally, the corresponding displacement field is shown in the diagrams of figure 5.b.

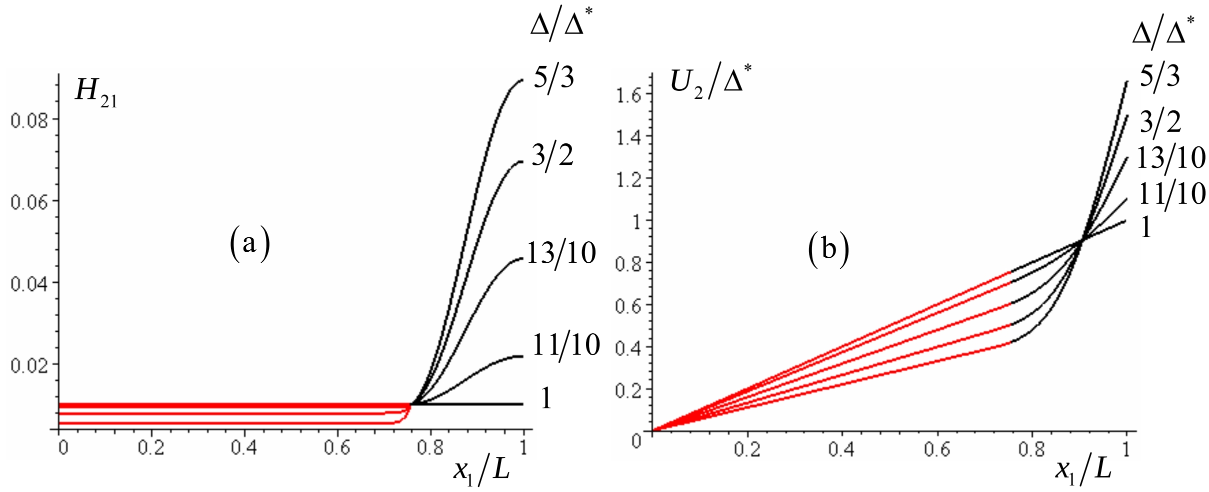

Figure 5. (a) Post-localization displacement gradient for varying prescribed displacement; (b) corresponding displacement field.

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