BUCKLING ANALYSIS OF VARIABLE ANGLE TOW COMPOSITE PLATES USING DIFFERENTIAL QUADRATURE METHOD

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Abstract
Variable Angle Tow (VAT) placement allows the designer to tailor the stiffness of composite structure to enhance the structural response under prescribed loading conditions. The Differential Quadrature Method (DQM) is investigated for performing buckling analysis of VAT panels. The governing differential equations are derived for the in-plane and buckling analysis of symmetric VAT plate based on classical laminated plate theory. DQM was applied to solve the buckling problem of simply supported VAT plates subjected to uniform edge compression and the results are compared with finite element analysis. In this work, Non-Uniform Rational B-Splines (NURBS) curves are used to model the fibre path by modifying the control points within the domain of the plate. Genetic algorithm (GA) has been coupled with DQM to determine the optimal tow path for improving the buckling performance.

1 Introduction
Conventional design of composite structures uses a standard combination of straight fibre laminae with constant thickness resulting in a constant stiffness structure and limited tailorability options. By allowing the fibre orientation to change in the plane of the structure, the stiffness of the structure can be varied resulting in redistribution of loads away from critical regions. This concept of VAT placement provides the designer a wider design space for tailoring the composite structure for enhanced structural performance under prescribed loading conditions. Hyer *et al.* [1] demonstrated the improvement of buckling resistance of composite plate with a hole using curvilinear fibre placement. Gurdal and his coworkers [2, 3] have demonstrated the superior buckling performance of VAT panels with linear fibre angle variation over straight fibre composites. Weaver *et al.* [4] employed an embroidery machine for tow steering and design of VAT plates with nonlinear angle distribution. Most of the works reported in literature for analysis of VAT panels use the finite element method (FEM) and when FEM is coupled with optimisation algorithms analysis becomes computationally expensive. To overcome this drawback, new methods are required which are fast, accurate and easily integrable with optimisation algorithms for design of VAT panels. In this work, numerical methodology based on the Differential Quadrature Method (DQM) is developed for buckling analysis of VAT panels. The stability and robustness of DQM in computing the buckling performance of VAT panels with linear and nonlinear fibre orientations are studied.

Different methods have been proposed in the literature to represent tow path variations in VAT plates. Gurdal *et al.* [2] introduced a linear fibre angle variation definition based on three fibre angle parameters (ϕ, T_0, T_1) which have subsequently been widely used for analysis, design and manufacture of VAT plates. This definition is simple and restricts the design space for tailoring the stiffness of VAT plates. Wu *et al* [5] used Lagrangian polynomial functions to represent the general fibre angle variation for VAT plates where the coefficients of polynomials are equal to the designed fibre angle at specified control points. The fibre angle based definition of tow path is suitable only for variation of fibre angle in a particular direction. When fibre angle based definition is applied to two dimensional variations, it becomes quite tedious to represent the tow path corresponding to the fibre angle distribution and also presents difficulties to manufacture them. The drawbacks of fibre angle variation can be overcome by using mathematical functions to represent directly the fibre path variation in the VAT plate. Nagendra *et al.* [6] used NURBS curves for tow path definition and they restricted their design of VAT plates to only five basic shape variations of NURBS curves for optimization studies. Parnas *et al.* [7] employed bi-cubic Bezier surfaces and cubic Bezier curves for layer thickness and fibre angle for design of the VAT plates. Kim *et al.* [8] used piecewise quadratic Bezier curve for defining the tow path in VAT plates. In the present work, the NURBS curves were used for tow path definition and this representation allows a general variation of fibre angle distribution across the x-y plane by moving the control points. This approach of tow path definition involves less design variables and the manufacturing constraints like minimum radius of curvature can be readily modeled.

The design of variable stiffness plates is more complex compared to straight fibre composites because of the number of design variables involved and the manufacturing constraints on designing continuous tow paths without any discontinuities for design of VAT plates. Ghiasi *et al.* [9] presented a good review of the different optimization algorithms for design of variable stiffness plates. Seetodeh *et al.* [10] used nodal based fibre angles as design variables and used a generalized reciprocal approximation technique for design of VAT plates for maximum buckling load. The limitations of this approach are the increase in number of design variables with mesh size and the difficulty in imposing fibre continuity across the plate structure. Lamination parameters were used by IJsselmuiden *et al.* [11] for designing VAT plates for maximizing the buckling load. This approach reduces the number of design variable, but requires considerable effort to construct manufacturable tow paths from the optimal lamination parameters. The design problem based on the proposed NURBS curve representation to determine the optimal fibre path for maximizing the buckling load is non-convex and genetic algorithm has been applied to solve it. The speed, efficiency and accuracy of the proposed DQM approach to solve buckling analysis of VAT plates reduce the computational effort required by GA to determine the optimal tow path.

2 Differential quadrature method

The differential quadrature method (DQM) was introduced by Bellman *et al*. [12] to solve initial and boundary value problems. In DQM, the derivative of a function, with respect to a space variable at a given discrete grid point is approximated as a weighted linear sum of function values at all of the grid points in the entire domain of that variable. The n^{th} order partial derivative of a function *f(x)* at the i^{th} discrete point is approximated by,

$$\frac{\partial^n f(x_i)}{\partial x^n} = A_{ij}^{(n)} f(x_j) \quad j = 1,2,\ldots N, \tag{1}$$

where x_i = set of discrete points in the x direction; and $A_{ij}^{(n)}$ is the weighting coefficients of the n^{th} derivative and repeated index j means summation from 1 to N. Detailed information regarding the grid distribution for computation of weighting coefficient matrices and handling multiple boundary conditions is given by Shu [13]. In this work, DQM is applied to solve the buckling problem of VAT plates under simply supported boundary conditions.

3 Analysis of VAT panels

In VAT panels, stiffness (A, B, D matrices) vary with x, y coordinates resulting in non-uniform in-plane stress distribution under constant edge loads or displacements [3]. A stress function formulation for in-plane analysis and displacement formulation for buckling analysis was employed to derive the governing differential equations (GDE) of VAT plates based on classical laminated plate theory.

3.1 Pre-buckling analysis

The GDE for in-plane analysis of symmetric VAT plate is given by,

$$\begin{aligned}
&A_{11}^{*}(x,y)\Omega_{,yyyy} - 2A_{16}^{*}(x,y)\Omega_{,xyyy} + \left(2A_{12}^{*}(x,y) + A_{66}^{*}(x,y)\right)\Omega_{,xxyy} - 2A_{26}^{*}(x,y)\Omega_{,xxxy} \\
&+ A_{22}^{*}(x,y)\Omega_{,xxxx} + \left(2A_{11,y}^{*}(x,y) - A_{16,x}^{*}(x,y)\right)\Omega_{,yyy} + (2A_{12,x}^{*}(x,y) - 3A_{16,y}^{*}(x,y) + \\
&A_{66,x}^{*}(x,y))\Omega_{,xyy} + (2A_{12,y}^{*}(x,y) - 3A_{26,x}^{*}(x,y) + A_{66,y}^{*}(x,y))\Omega_{,xxy} + (2A_{22,x}^{*}(x,y) - \\
&A_{26,y}^{*}(x,y))\Omega_{,xxx} + (A_{11,yy}^{*}(x,y) - A_{12,xx}^{*}(x,y) - A_{16,xy}^{*}(x,y))\Omega_{,yy} + (-A_{26,xx}^{*}(x,y) - \\
&A_{16,yy}^{*}(x,y) + A_{66,xy}^{*}(x,y))\Omega_{,xy} + (A_{12,yy}^{*}(x,y) - A_{22,xx}^{*}(x,y) + A_{26,xy}^{*}(x,y))\Omega_{,xx} = 0
\end{aligned} \tag{2}$$

where A^{*} is the in-plane compliance matrix and Ω is the Airy's stress function. Eqn 2 is a fourth order elliptic partial differential equation in terms of stress functions with variable coefficients and derivatives of compliance terms which represent the additional degrees of freedom available for tailoring of VAT plates when compared to straight fibre composites. The boundary conditions expressed using stress function for axial compression loading are given by

$$\begin{aligned}
&\left.\frac{\partial^2\Omega}{\partial y^2}\right|_{x=0,a} = \sigma_x(y),\ \left.\frac{\partial^2\Omega}{\partial x^2}\right|_{y=0,b} = 0,\ \left.\frac{\partial^2\Omega}{\partial x\partial y}\right|_{x=0,a;\,y=0,b} = 0, \\
&\Omega|_{x=0,y=b} = \left.\frac{\partial\Omega}{\partial x}\right|_{x=0,y=b} = \left.\frac{\partial\Omega}{\partial y}\right|_{x=0,y=b} = 0,
\end{aligned} \tag{3}$$

where $\sigma_x(y)$ is the applied compression loading along the edge $x = 0, a$ of the VAT plate.

3.2 Buckling analysis

The GDE for buckling analysis of symmetric VAT plate is given by,

$$\begin{aligned}
&D_{11}(x,y)w_{,xxxx}+4D_{16}(x,y)w_{,xxxy}+2\left(D_{12}(x,y)+2D_{66}(x,y)\right)w_{,xxyy}+4D_{26}(x,y)w_{,yyyx}+\\
&D_{22}(x,y)w_{,yyyy}+2\left(D_{11,x}(x,y)+D_{16,y}(x,y)\right)w_{,xxx}+(6D_{16,x}(x,y)+2D_{12,y}(x,y)+\\
&4D_{66,y}(x,y))w_{,xxy}+(2D_{12,x}(x,y)+4D_{66,x}(x,y)+6D_{26,y}(x,y))w_{,xyy}+2(D_{26,x}(x,y)+\\
&D_{22,y}(x,y))w_{,yyy}+(D_{11,xx}(x,y)+2D_{16,xy}(x,y)+D_{12,yy}(x,y))w_{,xx}+(2D_{16,xx}(x,y)+\\
&4D_{66,xy}(x,y)+2D_{26,yy}(x,y))w_{,xy}+(D_{12,xx}x(x,y)+2D_{26,xy}(x,y)+D_{22,yy}(x,y))w_{,xyy}\\
&+\overline{N}_x w_{,xx}+2\overline{N}_{xy}w_{,xy}+\overline{N}_y w_{,yy}=0
\end{aligned} \tag{4}$$

where D_{ij} is the laminate bending stiffness matrix and w is the out of plane displacement. The simply supported plate boundary conditions for the VAT plate are given by,

$$\begin{aligned}
&x=0,a;w=0;M_x=-D_{11}(x,y)\frac{\partial^2 w}{\partial x^2}-D_{12}(x,y)\frac{\partial^2 w}{\partial y^2}-2D_{16}(x,y)\frac{\partial^2 w}{\partial x\partial y}=0,\\
&y=0,b;w=0;M_y=-D_{12}(x,y)\frac{\partial^2 w}{\partial x^2}-D_{22}(x,y)\frac{\partial^2 w}{\partial y^2}-2D_{26}(x,y)\frac{\partial^2 w}{\partial x\partial y}=0.
\end{aligned} \tag{5}$$

The DQM is applied to numerically discretize the derivative terms in GDE and boundary conditions in Eqns 2, 3, 4 and 5 and they are reduced into a set of algebraic linear equations. The details regarding the numerical implementation of DQM for pre-buckling and buckling analysis of VAT plates can be found in this reference [16].

4 Buckling optimization formulation

In this work, GA is applied to determine the optimal tow path modeled using NURBS curves for maximizing the buckling load of VAT plates. The NURBS curve is defined by,

$$C(u)=\frac{\sum_{i=0}^{n}N_{i,p}(u)w_iP_i}{\sum_{i=0}^{n}N_{i,p}(u)w_i},\ a\le u\le b \tag{6}$$

where P_i are the control points, w_i are the weights and $N_{i,p}(u)$ are the p^{th} degree B-spline functions defined on the non-uniform knot vector. The design of the tow path is done using a NURBS curve passing through a set of control points as shown in Fig 1. The fibre angle distribution is computed from the tow path and is given as input for DQM buckling analysis of VAT plates. The control points define the shape of the tow path and the co-ordinates of the control points are taken as the design variables in this formulation. The optimization problem can be cast in the mathematical programming form given by,

Maximize: Buckling coefficient K_{cr},
Design Variables: $\{X_{11}, X_{12},......, X_{1n}, X_{2n}\}$, n is the number of control points,
Subject to: $X^L \le X_{1i}, X_{2i} \le X^U$ (bounds on design variables). The optimal tow path results obtained using GA is presented in the next section.

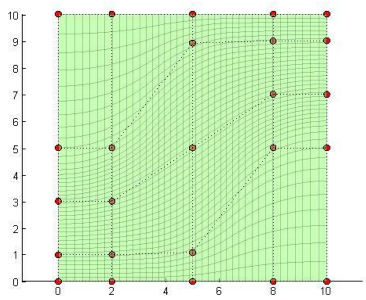


Figure 1. NURBS representation of tow path (red dots are the control points for shape control)

5 Results and discussion

5.1 Model validation

The pre-buckling and buckling results obtained using DQM are presented in this section for VAT plates with linear fibre angle variation for validation purposes. The material properties for each lamina are chosen as, E_1=181GPa, E_2= 10.27GPa, G_{12}= 7.17GPa, ν_{12} = 0.28 with thickness t=0.1272 mm. FE modelling of the VAT panels was carried out using ABAQUS. The S4 shell element was chosen for discretization of the VAT plate structure and appropriate mesh size was selected to achieve the desired accuracy. A square VAT plate (a = b = 1m) subjected to uniform axial compression along the edges x = 0; a and other stress boundary conditions are shown in Fig.2. A symmetric VAT plate $[0\pm\langle 0|45\rangle]_{3s}$ was chosen for the in-plane analysis and the number of grid points for DQM modelling was chosen to be $N_x = N_y = 30$.

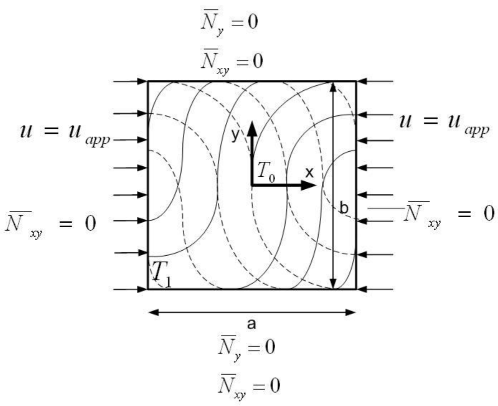


Figure 2. Square VAT plate subjected to uniform compression

The stress resultant distributions obtained using DQM and FEM are shown in Fig. 3 and are close to each other. Subsequently DQM was applied to perform buckling analysis of square VAT plates under simply supported boundary condition with fibre orientation perpendicular to the loading direction $[90\pm\langle T_0|T_1\rangle]_{3s}$. The non-dimensional buckling coefficient and non-dimensional stiffness of VAT plates are evaluated by the following relation,

$$K_{cr} = \frac{N_{vat}a^2}{Ebh^3},\; E_{vat} = \frac{a\int_0^b N_x(a,y)dy}{bhu_{app}}, \tag{7}$$

where N_{vat} is the lowest critical buckling load of VAT plate. Non-dimensional values of buckling coefficient versus stiffness for various VAT plate configurations $[90\pm\langle T_0|T_1\rangle]_{3s}$ obtained using DQM and FE method are shown in Fig. 4. The results match very well and clearly shows the variation in buckling load for different values of T_0, T_1 with maximum value achieved for the VAT plate configuration $[90\pm\langle 0|80\rangle]_{3s}$.

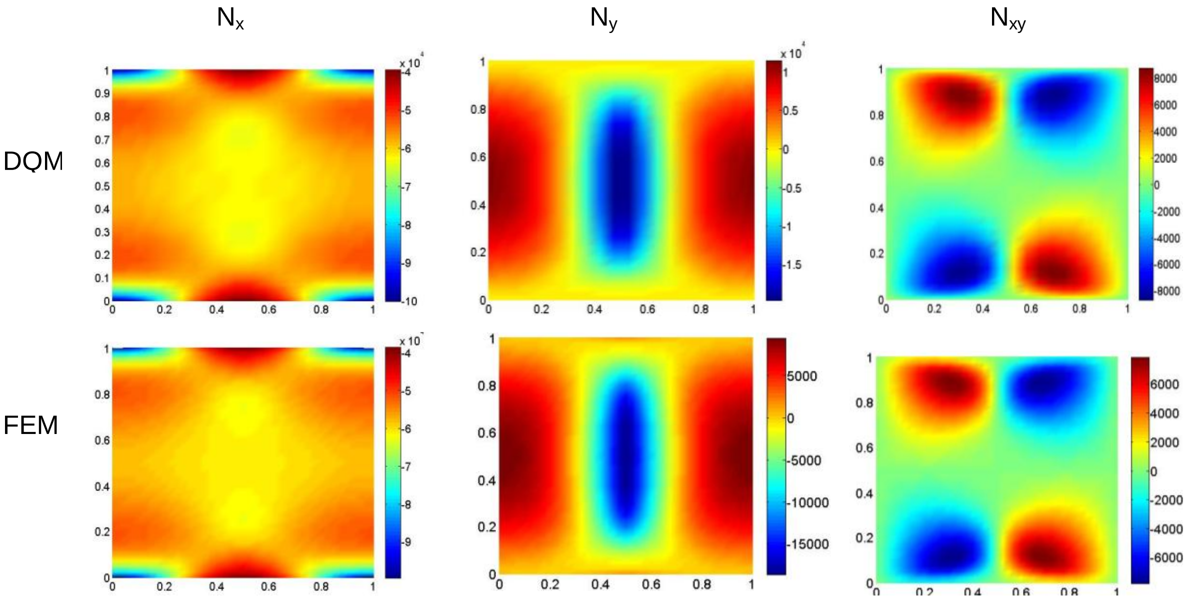


Figure 3. DQM and FEM in-plane stress resultant distributions (N_x, N_y, N_{xy}) results for a square VAT plate $[0\pm\langle 0|45\rangle]_{3s}$ subjected to uniform axial compression.

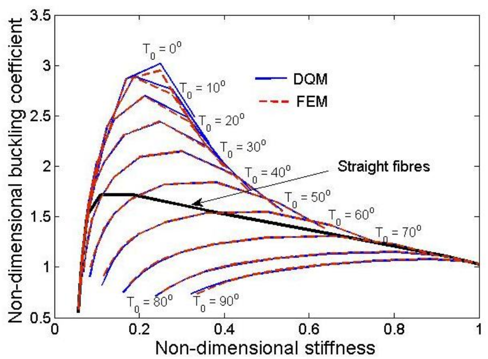
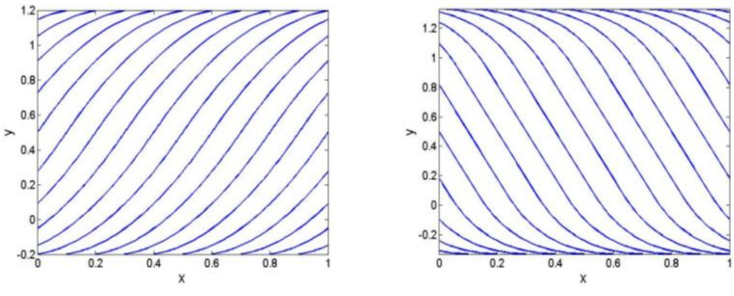


Figure 4. Square VAT plate $[90\pm\langle T_0|T_1\rangle]_{3s}$ subjected to axial compression and simply supported plate boundary conditions: Non-dimensional buckling coefficient versus Non-dimensional stiffness.

5.2 Optimisation results

A square symmetric and balanced VAT laminate ($[\pm\theta_1/\pm\theta_2]_s$) subjected to axial compression and simply supported boundary conditions is considered in the optimisation study. For modelling and optimisation of the tow path, NURBS curves of order 3 and 7 control points were used. To observe the redistribution of the applied compression loading in the VAT plate, the change in fibre angle variation of the tow path was restricted to be perpendicular to the loading direction (y direction). The population size and number of generations for GA optimisation were chosen to be 25 and 100 respectively. The optimal tow paths obtained for both the VAT laminas are shown in Fig. 5 and it illustrates the smooth and continuous variation of tow paths in the plate. Figure 6 shows the variation of buckling coefficient with generations and GA converges to the optimal results after 30 generations. The buckling coefficient computed using DQM for the optimal VAT plate configuration is K_{cr} = 3.5026 and it shows an increase of 16.75% over the optimal VAT plate with linear fibre angle variation [3]. The axial stress resultant distribution N_x and the buckling mode shape for the optimal VAT plate design is shown in Fig. 7 and redistribution of the applied compressive load from the centre of the plate toward the edges was observed.



(a) (b)

Figure 5. GA optimal tow path results: (a) VAT layup 1 (b) VAT layup 2.

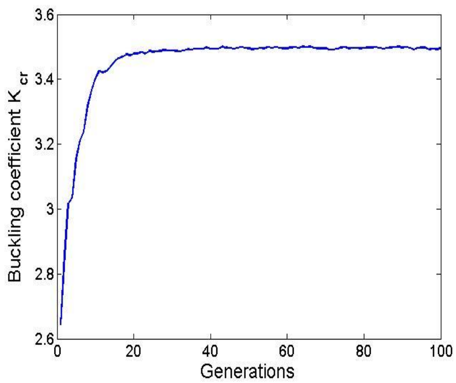
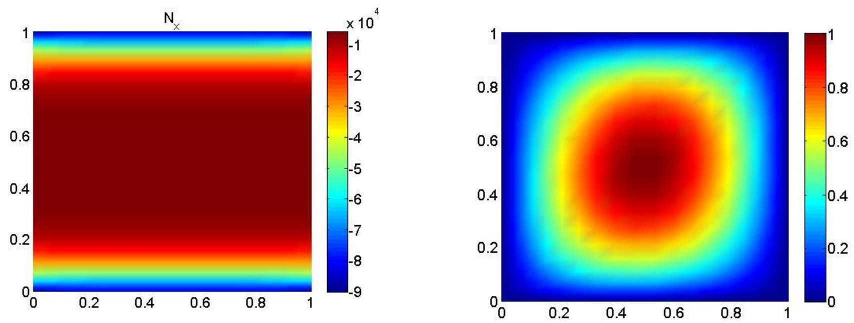


Figure 6. GA optimization results: Buckling coefficient (K_{cr}) versus Generations.



(a) (b)

Figure 7. Stress distribution and buckling results for optimal VAT plate ($[\pm\theta_1/\pm\theta_2]_{2s}$) (Fig. 5) (a) N_x distribution (b) Buckling mode shape (K_{cr} = 3.5026).

6. Conclusions

DQM was successfully applied to solve the buckling analysis problem of symmetric VAT composite panels with simply supported boundary conditions. The results obtained using

DQM compared very well with FEM results. This study shows the stability and robustness of DQM in solving the buckling analysis of VAT plates with general fibre angle orientations. A NURBS representation for modelling of two paths is presented and this allows an extendable design space for VAT plates with general fibre angle distributions. GA was coupled along with DQM to determine the optimal tow path for maximising the buckling load of VAT plates. The design results shows that the integration of DQM with GA is robust, efficient and requires relatively less computational effort. In future, the proposed optimisation strategy can be extended for designing of VAT plates for maximising the buckling loads under shear and combined loading conditions.

References

[1] Hyer M.W., Lee H.H. The use of curvilinear fibre format to improve buckling resistance of composite plates with central circular holes. *Compos Struct*, **18**, 239-61 (1991).

[2] Gurdal Z., Olmedo R. In-plane response of laminates with spatially varying fibre orientations: variable stiffness concept. *AIAA J.*, **31(4)**, 751-58 (1993).

[3] Gurdal Z., Tatting B.F., Wu C.K. Variable stiffness composite panels: Effect of stiffness variation on the in-plane and buckling response. *Compos Part A*, **39**, 911-22 (2008).

[4] Weaver P.M., Potter K.D., Hazra K, Saverymuthupulle M.A.R., Hawthorne M. T. *Buckling analysis of variable angle tow plates: from concept to experiment* in " Proceeding of *50th AIAA/ASME/ASCE/AHS/ ASC Structures, Structural dynamics and Materials conference*", California, USA, (2009).

[5] Wu Z., Raju G., Weaver P.M. Buckling analysis and optimization of variable angle tow plates, *submitted to Thin-walled structures* (2012).

[6] Nagendra S, Kodilyam S.,Davis J.E. *Optimization of tow fibre paths for composite design* in " Proceeding of *36th AIAA/ASME/ASCE/AHS/ ASC Structures, Structural dynamics and Materials conference*", Los Angeles, USA, (1995).

[7] Parnas L., Oral S., Ceyhan U., Optimum design of composite structures with curved fibre courses, *Comp. Sci. Tech.*, **63 (7)**, 1071-82 (2003).

[8] Kim B.C., Potter K.D., Weaver P.M., Continuous tow shearing for manufacturing variable tow composites, *Compo Part A (accepted)*, (2012).

[9] Ghiasi H., Fayazbakhsh K., Pasini D., Lessard L., Optimum stacking sequence design of composite materials Part II: Variable stiffness design, *Compos Struct* , **93**, 1-13 (2010).

[10] Seetodeh M., Abdalla M.M., IJsselmuiden S.T., Gurdal Z. Design of variable stiffness composite panels for maximum buckling load, *Compos Struct*, **87**, 109-17 (2008).

[11] IJsselmuiden S.T., Abdalla M.M., Gurdal Z. Optimization of variable stiffness panels for maximum buckling load using lamination parameters, *AIAA J.*, **48(1)**, 134-143 (2010).

[12] Bellmann R.E., Casti. J. Differential quadrature and long-term integration, *J Math Anal Appl*, **34**, 235-38, 1971.

[13] Shu C., *Differential quadrature and its application in engineering*, Springer-verlag, London, 2000.

[14] Raju G., Wu Z., Kim B.C., Weaver P.M., Prebuckling and buckling analysis of variable angle to plates with general boundary conditions. *Compos Struct (accepted)*, (2012).