EFFECT OF THE SHEAR CORRECTION FUNCTION IN THE FGM BEAMS MODAL ANALYSIS

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Abstract

In the proposed contribution, effect of the shear correction function will be originally studied and evaluated in the modal FGM beams analysis. Spatial continuous variation of the beam material properties has been considered. The shear correction function will be calculated from the shear strain energy equation including the spatial Poisson's ratio variation. The four coupled differential equations have been derived and used in the modal analysis of the FGM beams. The 2nd order beam theory and the longitudinal varying Winkler elastic beam foundation will be considered.

1 Introduction

For homogeneous material properties of the whole beam the shear correction factor is a constant and depends on geometry of the beam cross-sectional area. Using the expression for quadratic shear stress distribution for calculation of the shear strain energy and putting it equal to the shear strain energy of the first order shear deformation theory, the shear correction factor can be calculated [1], [2], [3]. As shown in [4], the shear force deformation effect influences also the behavior of sandwich and composite beams. Similar effect occurs also in the Functionally Graded Material (FGM) beams with continuous or discontinuous transversal or spatial variation of material properties. In [5], the shear force deformation effect in FGM beams has been originally described using the shear correction function. If the material properties change in longitudinal beam axis the shear force deformation effect also changes in this direction. The average value of the shear correction function (the average shear correction factor) has been calculated and implemented into the transfer relations for deformation and modal analysis of the FGM beams. Using of the average shear correction factor makes the calculation simpler, but a stronger longitudinal non-linear distribution of the shear deformation effect can affect the calculated results in negative sense remarkable. A constant Poisson's ratio has been used in the most considerations of the shear force deformation effect.

The four coupled differential equations of the homogenized functionally graded material (FGM) beam deflection and their solution will be presented in the proposed contribution which will be used in modal analysis of the beams with polynomial continuous longitudinal and continuous symmetric transversal variation of material properties. The 2nd order beam theory and the longitudinal varying Winkler elastic beam foundation will be considered. Effect of the shear correction function (without its averaging) will be originally studied and evaluated. The shear correction function will be calculated from the shear strain energy equation, where the non-constant Poisson's ratio will be considered.

2 Shear force correction function derivation

Several shear theories are known, which differ each to other with degree of the beam cross-section deformation. As has been shown in the literature, the shear force effect is also not negligible in the analysis of the composite structures which are made as sandwich or multilayer structures [6], [7]. Similar effect has been also shown by FGM structures [5], [8] which are made of continuous or discontinuous variation of material properties. By the transversal variation of material properties the shear force effect is constant for whole beam, but its value differ from the standard value known from the shear theory of beams made of homogenous material. The ideal shear force stiffness is reduced with the shear correction factor in the 2nd order shear deformation theory. If the material properties vary in longitudinal direction the shear correction factor vary also in this direction. The shear correction function has to be considered.

Let us consider a two nodal straight beam element with constant rectangular cross-sectional area $A = bh$ and quadratic moment of inertia $I = bh^3/12$ (Figure 1).

The FGM of this beam arises from mixing two components (formally named as matrix and fibres) that are approximately of the same geometrical form and dimensions. The continuous spatial variation of the effective material properties can be caused by continuous spatial variation of both the volume fraction and material properties of the FGM constituents. Both the fibers volume fraction $v_f(x, y)$ and the matrix volume fraction $v_m(x, y)$ are chosen as a polynomial functions of *x*, and with continuous and symmetrical variation through its height *h* with respect to the neutral plane of the beam. The volume fractions are assumed to be constant through the cross-section depth *b*. At each point of the beam it holds: $v_f(x, y) + v_m(x, y) = 1$.

The values of the volume fractions at the nodal points are denoted by indices *i* and *j*. The assumption of polynomial variation enables an easier establishing of the beam equations and allows modeling many common continuous variations of beam parameters.

The material properties of the constituents (fibres - $p_f(x, y)$ and matrix - $p_m(x, y)$) vary analogically as stated by the variation of the volume fractions. For effective material property $p(x, y)$ in the real beam we have got:

$$p(x, y) = v_f(x, y) p_f(x, y) + v_m(x, y) p_m(x, y). \tag{1}$$

In our case the elasticity modulus $E(x, y)$, Poisson ratio $\nu(x, y)$, and mass density $\rho(x, y)$ have been calculated by expression (1). The FGM shear modulus can be calculated by expression:

$$G(x, y) = \frac{E(x, y)}{2(1 + \nu(x, y))}. \tag{2}$$

If the constituents Poisson's ratio are approximately of the same value and the constituents' volume fraction variation is not strong, then the FGM shear modulus can by calculated using a simplification:

$$G(x,y) = \frac{E(x,y)}{\xi}, \tag{3}$$

where ξ is an average value of the function $\xi(x,y) = 2(1+\nu(x,y))$

$$\xi = \frac{1}{L}\int_0^L \left(\frac{1}{h} \int_{-h/2}^{h/2} \xi(x,y)dy \right) dx. \tag{4}$$

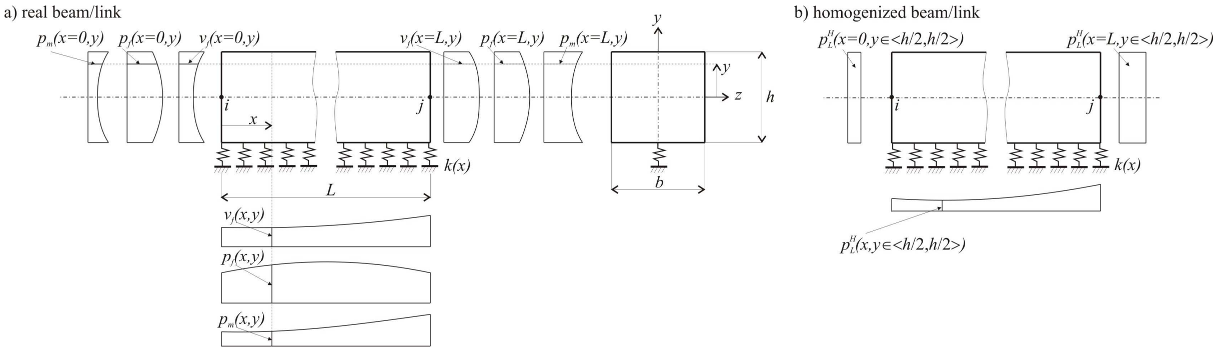

Figure 1. Real and homogenized FGM beam element.

The direct integration method [9] will be used for the homogenization of the spatial varying material properties (1). From the assumption that the respective property (e.g. stiffness) of the real beam must be equal to the analogical property of the homogenized beam, the homogenized longitudinal elasticity modules for: tension – compression $E_L^{NH}(x)$, bending $E_L^{MH}(x)$, shear $G_L^H(x)$, and the homogenized mass density $\rho_L^H(x)$, can be calculated [14], respectively. The homogenized material properties have been used in establishing the shear correction function and the differential equations of free vibration of the homogenized FGM beam.

Figure 2 shows character of the shear stress distribution in the rectangular cross section at the position *x*.

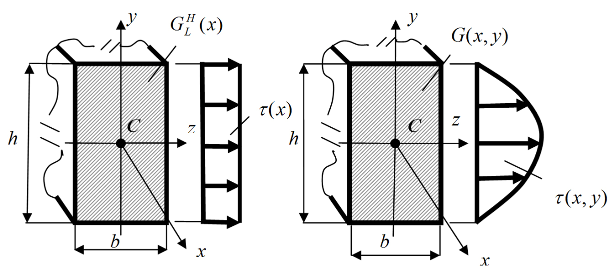

Figure 2. Shear stress distribution.

The shear stress $\tau(x,y)$ in the cross section of real beam has symmetric non-linear distribution over the beam height (the non-linearity depends of the $E(x,y)$ variation) [9]:

$$\tau(x,y) = \frac{Q(x)}{B_I(x)} \int_y^{h/2} E(x,y)y\,dy \tag{5}$$

where $B_I(x)=\int_{-h/2}^{h/2} by^2 E(x,y)\,dy$ is the effective bending stiffness of the real FGM beam, and $Q(x)$ is the shear force at position *x*. The shear stress $\tau(x)$ in the homogenized beam is assumed as constant over the cross-section:

$$\tau(x)=\frac{Q(x)}{A} \tag{6}$$

Using the expression (5) for the calculation of the strain energy in the cross-sectional area of the real FGM beam and putting it equal to the strain energy in the homogenized beam of the first order shear deformation theory, the shear correction function $k^s(x)$ can be calculated:

$$k^s(x)=\frac{\int_A \frac{\tau(x)^2}{G_L^H(x)}dA}{\int_A \frac{\tau(x,y)^2}{G(x,y)}dA} \tag{7}$$

The integration in (7) has been solved with software MATHEMATICA [10]. Some numerical problems can occur in computation of $k^s(x)$ due to by complicated variations of the shear modulus $G(x,y)$. The shear correction function $k^s(x)$ will later input into the free vibration equations of the FGM beam through the stiffness ratio function:

$$\varsigma(x)=\frac{E_L^{MH}(x)I}{k^s(x)G_L^H(x)A} \tag{8}$$

An average shear correction factor:

$$k^{sm}=\frac{1}{L}\int_0^L k^s(x)dx \tag{9}$$

can be, under some circumstances, considered in calculation of the parameter (7) that can be used in simpler but less accurate modal analysis of the FGM beams.
If the shear force effect has been neglected, the ratio function $\varsigma(x)$ is equal to zero. The above presented method can be also applied for other cross-section types.

3 Mathematical base of the differential equations derivation

According to [13], the main equations of the 2nd order beam theory (including the inertia forces) are:

$$R'=-q+kw-\mu\omega^2 w \tag{10}$$

$$M'=Q+m+\bar{\mu}\omega^2\varphi \tag{11}$$

$$\varphi'=-\frac{M}{EI} \quad \Rightarrow \quad M=-EI\varphi'-EI\kappa^e \tag{12}$$

$$w'=\varphi+\frac{Q}{G\tilde{A}} \quad \Rightarrow \quad Q=G\tilde{A}w'-G\tilde{A}\varphi \tag{13}$$

Here, *q* is the distributed transversal load (see Figure 3); *m* is the distributed bending moment; κ^e is he applied beam curvature; *k* is the modulus of elastic Winkler foundation; μ is the mass distribution; $\bar{\mu}$ is the mass inertial moment distribution; ω is the natural eigenfrequency; R is the transversal force; Q is the shear force; M is the bending moment; φ is the angle of cross-section rotation; w is the beam bending; EI is the bending stiffness and $G\tilde{A}$ is the reduced shear stiffness of the homogenized FGM beam. We assume that all the above quantities are the polynomial functions of x.

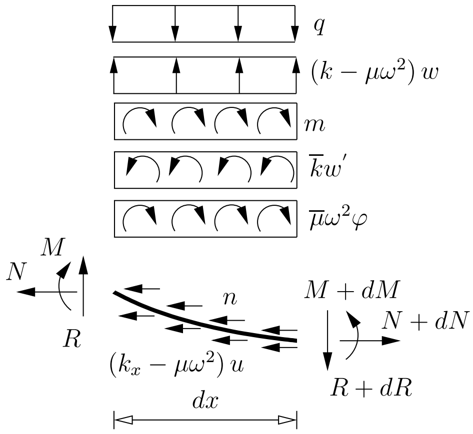

Figure 3. The force equilibrium in the deformed element configuration.

The relation between the transversal and shear force is:

$$Q = -(\bar{k} + N^{II})w' - N^{II}\psi + R \tag{14}$$

where $N^{II} \equiv N$ is the resultant axial force of the 2nd order beam theory, ψ is the beam rotation imperfection, and $\bar{k}$ is the elastic foundation modulus for beam rotation. The derivation of the four coupled differential equations and their solution for the buckling force and eigenfrequency will be described in [14] in detail.

4 Numerical experiments

Cantilever beam (Fig. 4) is made of a mixture of titanium carbide TiC (fibres) – the elasticity modulus $E_f = 480.0$ GPa, the mass density $\rho_f = 4920$ kgm^{-3}, the Poisson's ratio $\nu_f = 0.20$; and aluminum Al 6061-TO (matrix) – the elasticity modulus $E_m = 69.0$ GPa, the mass density $\rho_m = 2700$ kgm^{-3}, the Poisson's ratio $\nu_m = 0.33$; [11]. Its geometry is given with: $b = h = 0.01$m, $L = 0.1$m.

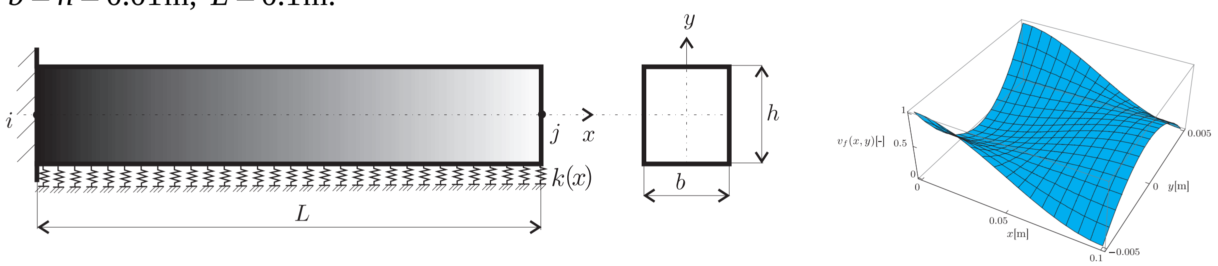

Figure 4. Cantilever beam with planar variation of material properties.
Volume fraction variation - planar variation along the beam length and beam height.

Variation of the fibres volume fraction has been chosen as the polynomial function:

$$v(x,y) = \frac{400000000x^3y^2}{3} - \frac{400x^3}{3} - 20000000x^2y^2 + 200x^2 + 40000y^2$$

that is drawn in Figure 4 – left side.
Using the extended mixture rule (1), (2), a spatial distribution of the effective elasticity modulus $E(x,y)$ in [GPa], the Poisson's ratio $\nu(x,y)$ [-], shear modulus $G(x,y)$ in [GPa] and mass density $\rho(x,y)$ in [kgm^{-3}] have been calculated:

$$E(x,y) = 548\times10^{8}x^{3}y^{2} - 548000x^{3} - 822\times10^{7}x^{2}y^{2} + 82200x^{2} + 16440000y^{2} + 69 \quad (15)$$

$$\nu(x,y) = -1.7334\times10^{7}x^{3}y^{2} + 173.3334x^{3} + 2.6\times10^{6}x^{2}y^{2} - 26.0x^{2} - 5200.0y^{2} + 0.33 \quad (16)$$

$$\begin{aligned} G(x,y) &= (548\times10^{8}x^{3}y^{2} - 548000x^{3} - 822\times10^{7}x^{2}y^{2} + 82200x^{2} + 16440000y^{2} + 69) \\ &/(2.66 - 3.466666666\times10^{7}x^{3}y^{2} + 346.6666666x^{3} + 5.2\times10^{6}x^{2}y^{2} - 52.0x^{2} - 10400.0y^{2}) \end{aligned} \quad (17)$$

$$\rho(x,y) = 269\times10^{9}x^{3}y^{2} - 296\times10^{4}x^{3} - 444\times10^{8}x^{2}y^{2} + 444000x^{2} + 888\times10^{5}y^{2} + 2700 \quad (18)$$

The effective beam properties have been calculated using the expressions derived in [14], it was obtained:

$$E_L^{NH}(x) = -9.133333333\times10^{10}x^{3} + 1.37\times10^{10}x^{2} + 2.06\times10^{8} \text{ [kPa]} \quad (19)$$

$$E_L^{MH}(x) = 2.74\times10^{11}x^{3} - 4.11\times10^{10}x^{2} + 3.156\times10^{8} \text{ [kPa]} \quad (20)$$

$$\begin{aligned} G_L^{H}(x) &= 8.15909138\times10^{7} + 8.552673524\times10^{6}x + 3.187975901\times10^{9}x^{2} \\ &+ 6.430131676\times10^{9}x^{3} - 2.116329771\times10^{11}x^{4} \end{aligned} \text{ [kPa]} \quad (21)$$

$$\rho_L^{H}(x) = -4.933333333\times10^{5}x^{3} + 74000.0x^{2} + 3440.0 \text{ [kgm}^{-3}\text{]} \quad (22)$$

The shear stress distribution (5) at several position of x and $Q(x)=1$, are:

$$\tau(x=0,y) = 13046.57 - 1.56273\times10^{13}y^{4} - 1.311787072\times10^{8}y^{2} \text{ [kPa]} \quad (23)$$

$$\tau(x=0.5L,y) = 14584.17 - 3.326588425\times10^{12}y^{4} - 5.002023471\times10^{8}y^{2} \text{ [kPa]} \quad (24)$$

$$\tau(x=L,y) = 17301.23 + 1.840985443\times10^{13}y^{4} - 1.152295633\times10^{9}y^{2} \text{ [kPa]} \quad (25)$$

which are drawn in Figure 5 – left side.

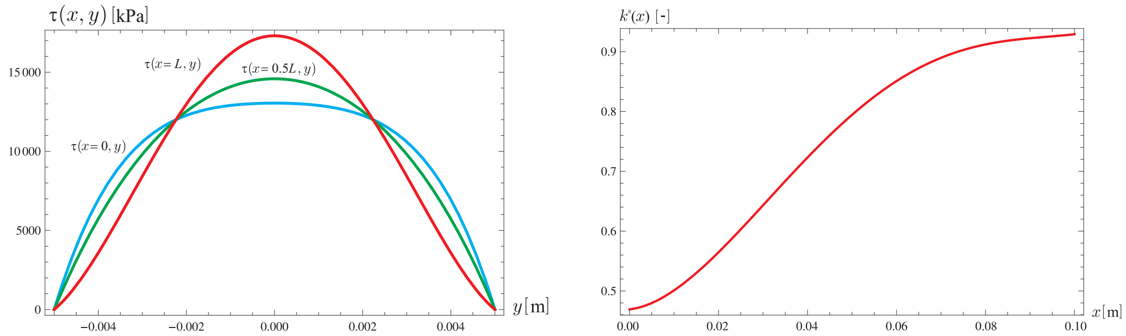

Figure 5. Distribution of the shear stresses (24), (25), (26), the shear correction function.

The homogenized bending stiffness used in the shear stress calculation (5) is:

$$B_I(x) = 228.34x^{3} - 34.25x^{2} + 0.263 \text{ [kNm}^{2}\text{]} \quad (26)$$

The shear correction function calculated with (7) is:

$$k^{s}(x) = 0.469249 + 0.906959x + 262.441x^{2} - 3766.23x^{3} + 15109.2x^{4} \quad (27)$$

Longitudinal distribution of the shear correction function is shown in Figure 5 – right side, its value is 0.469 at point *i* and 0.929 at point *j*. The average shear correction factor (9) is: k^{sm} = 0.75. The FGM cantilever beam (Figure 4) clamped at the left side and resting on the Winkler elastic foundation has been studied by modal analysis. Strong Winkler elastic foundation modulus has been chosen as a varying non-linear function of *x*: $k(x) = 5000 - 1000x + 6000x^2$ [kN/m^2]. The first three bending eigenfrequencies have been found using the differential equations (10) – (13) for 3 options of the shear force deformation effect consideration: without the shear effect - $\varsigma(x) = 0$; with the average shear correction factor - $k^{sm} = 0.75$; and with the shear correction function - $k^s(x)$. The same problem has been solved using a very fine mesh – 12000 of 2D PLANE42 elements of the FEM program ANSYS [12], where the material properties (15) - (18) have been used. The average relative difference Δ[%] (for three chosen values of *N* force of each) between eigenfrequencies calculated by our method and the ANSYS solution has been evaluated.
To show the effect of normal force *N* on the eigenfrequency, its positive value (tension), negative value (compression), and equal to zero have been taken into account. The nonzero values of the axial force have been chosen by $N \cong \pm 0.75 N_{Ki}^{II}$ in all cases calculated, where N_{Ki}^{II} is the 2nd order beam theory buckling force. The buckling force has been calculated from the differential equations by setting $\omega = 0$. Tables 1, 2 and 3 contain the 1st, 2nd and 3rd bending eigenfrequency.

N [kN] Option:	52	0	-52	Δ [%]	N_{Ki}^{II} [kN]
$\varsigma(x) = 0$	2 048.9	1 599.2	808.4	1.35	-67.71
$k^{sm} = 0.75$	2 031.5	1 585.8	797.0	0.30	-67.24
$k^s(x)$	2022.3	1 582.8	797.7	0.11	-67.31
ANSYS	2 019.5	1 582.6	796.2		

Table 1. The 1st eigenfrequency.

N [kN] Option:	52	0	-52	Δ [%]	N_{Ki}^{II} [kN]
$\varsigma(x) = 0$	9 133.8	8 432.4	7 654.7	9.23	-67.71
$k^{sm} = 0.75$	8 719.5	8 018.9	7 238.4	3.81	-67.24
$k^s(x)$	8 556.1	7 865.3	7 092.5	1.80	-67.31
ANSYS	8 399.6	7 727.2	6 970.0		

Table 2. The 2nd eigenfrequency.

N [kN] Option:	52	0	-52	Δ [%]	N_{Ki}^{II} [kN]
$\varsigma(x) = 0$	23 301.6	22 687.3	22 056.0	20.46	-67.71
$k^{sm} = 0.75$	21 087.4	20 447.2	19 791.2	8.56	-67.24
$k^s(x)$	20 460.5	19 820.2	19 171.3	5.24	-67.31
ANSYS	19 418.0	18 838.0	18 235.0		

Table 3. The 3rd eigenfrequency.

The eigenmodes for all the considered options (Table 1 to Table 3) have been calculated which will be presented in [14] in detail.

5 Conclusion

The shear correction function of the second order shear deformation theory has been derived from the shear strain energy equation. This has been then used in the four coupled differential equations derivation, which were used in modal analysis of the FGM beams. The effect of axial force and varying Winkler foundation has been considered.

The obtained solution results in the Table 1 to Table 3 induce following general conclusions: the average difference between our and the benchmark ANSYS eigenfrequency solution increases with the eigenfrequency number; the best agreement of both results is for the 1st eigenfrequency; the shear force effect is meaningful in all calculated cases; the most accurate results have been obtained by consideration the shear correction function $k^s(x)$ in the eigenfrequency calculation; using the average shear correction factor makes the shear effect calculation simpler but the results are of love accuracy comparing to the solution with the shear correction function consideration; inconsistent consideration of the Poisson's ratio variation (as a constant for example) can affect the solution results by the FGM beams.

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References

[1] S. P. Timoshenko, J. N. Goodier *Theory of elasticity*. 3rd ed. New York: McGraw- Hill, 1984.

[2] C. M. Wang, J. N. Reddy, K. H. Lee *Shear deformable beams and plates. Relationships with classical solutions*. Elsevier Science Ltd., ISBN 0-08-043784-2, 2005.

[3] H. Mang, G. Hofstetter *Festigkeitslehre*. Springer-Verlag, 2000.

[4] H. Altenbach, J. Altenbach, W. Kissing, *Mechanics of Composite Structural Elements. Engineerig* - Monograph, Springer-Verlagg, 463 p.,2003.

[5] J. Murín, M. Aminbaghai, V. Kutiš Exact solution of the bending vibration problem of FGM beams with variation of material properties. *Engineering Structures*, **32**, 1631-1640, 2010.

[6] E.J. Sapountzakis, J.A. Dourakopoulos, Shear deformation effect in flexural-torsion vibrations of composite beams by boundary element method. *Journal of vibration and control*, **16**, issue 12, 1763-1789, 2010.

[7] E.J. Sapountzakis, A.J. Dourakopoulos *Nonlinear dynamic analysis of Timoshenko beams*. Computational methods in earthquake engineering Book Series: Computational methods in applied sciences, **21**, 377-400, 2011.

[8] M. Aminbaghai, J. Murín, V. Kutiš Modal Analysis of the FGM-beams with Continuous Transversal Symmetric and Longitudinal Variation of Material Properties with Effect of Large Axial Force. *Engineering Structures*, **34**, 314-329, 2012.

[9] V. Kutiš, J. Murín, R. Belák, J. Paulech Beam element with spatial variation of material properties for multiphysics analysis of functionally graded materials. *Computers and Structures*, **89**, 1192 – 1205, 20011.

[10] S. Wolfram MATHEMATICA 5, Wolfram research, Inc., 2003.

[11] L. Banks-Sills, R. Eliasi, Y. Berlin Modeling of functionally graded materials in dynamic analyses. *Composite Part B: Engineering*, **33**, Issue 1, 7-15, 2002.

[12] ANSYS Swanson Analysis System, Inc., 201 Johnson Road, Houston, PA 15342/1300, USA.

[13] H. Rubin, *Analytische Berechnung von Stäben und Stabwerken mit stetiger Veränderlichkeit von Querschnitt, elastischer Bettung und Massenbelegung nach Theorie erster und zweiter Ordnung*, Baustatik - Baupraxis 7. Berichte der 7. Fachtagung "*Baustatik - Baupraxis*" Aachen/Deutschland 18.-19. März 1999. Hrsg.: Balkema 1999, Abb.,Tab.S.135-145

[14] J. Murin, M. Aminbaghai, J. Hrabovský, V. Kutiš, S. Kugler, The FGM beam modal analysis with effect of the shear correction function. In preparation for publication, 2012.