SHAKEDOWN LIMIT IN BRITTLE-MATRIX COMPOSITES WITH PLASTIC CRACK BRIDGING FIBERS UNDER COMBINED TRACTION-FLEXURE

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Abstract

The behaviour of a composite beam with multiple reinforcing fibers under periodic traction-flexure is analysed through a fracture mechanics-based model. In more detail, an edge-cracked beam under external loads is also subjected to the crack bridging reactions due to the fibers. Assuming a rigid-perfectly plastic bridging law for the fibers and a linear-elastic law for the matrix, the statically indeterminate bridging forces are obtained from compatibility conditions. Under assigned load paths, shakedown conditions are explored by making use of the Melan's theorem, here reformulated for the discrete problem under consideration, where crack opening displacement at the fiber level plays the role of plastic strain in the counterpart problem of an elastic-plastic solid. The limit of shakedown is determined through an optimization procedure based on a linear programming technique.

1 Introduction

Several composite materials used in different engineering applications consist of a brittle matrix and ductile reinforcements (bars, wires, fibers, etc.). By incorporating such reinforcements into the matrix, several mechanical properties are enhanced, including: cracking resistance, ductility, impact resistance, fatigue strength. Cracks might develop in structures of reinforced brittle-matrix composites, so that the overall mechanical behaviour, including the capacity to dissipate energy under cyclic loading, would strongly be affected by the crack bridging reactions of the reinforcements. Moreover, the progressive crack growth under cyclic loading influences the bridging behaviour, and causes significant changes in the mechanical properties of the above materials (strength, toughness, stiffness, hysteretic behaviour, etc.), eventually leading to failure.

Numerous theoretical models are available in the literature to describe the crack bridging behaviour of fiber-reinforced composites. For instance, under monotonic loading, the mechanics of elastic fibers, which might debond at the fiber-matrix interfaces, is investigated in Refs [1-4] with reference to their bridging effect on matrix cracking. Under periodic loading, the crack bridging behaviour, including cyclic debonding at fiber–matrix interface of fibers, is analysed in Refs [5-9] with the aim of predicting also the fatigue strength of the composite materials. According to the model proposed by the first two authors in Refs [10, 11] (see also Ref. 12), a fibrous composite beam with an edge crack submitted to cyclic bending moment can be examined by assuming a crack bridging model with a general linear isotropic tensile softening/compressive hardening law for the fibers and a linear- elastic law for the matrix. Elastic and plastic shakedown phenomena can be discussed in terms of generalized cross-sectional quantities and, by employing a fatigue crack growth law, the mechanical behaviour up to failure can be captured.

Within the framework of the LEFM-based model proposed in Refs [10, 11], the present paper is devoted to investigate, under combined axial force and bending moment describing general periodic load paths, the conditions of (elastic) shakedown (in the following the plain word 'shakedown' is used to mean 'elastic shakedown') by exploiting the Melan's theorem [13]. As a matter of fact, a parallel between the classical problem of an elastic-perfectly plastic body, for which the Melan's theorem was originally formulated, and the present crack bridging model with rigid-plastic fibers is drawn in the following. Then, the limit condition of shakedown under any traction-flexure history within a given load domain is determined through an optimization procedure.

2 The crack bridging model

Consider an edge-cracked portion of fiber-reinforced composite beam with a rectangular cross-section under time-varying axial force *F(t)* and bending moment *M(t)* (Fig.1), where time t should be regarded as ordering variable of the events, being the problem under consideration nominally static. The crack (which might possibly be regarded as an existing flaw) in the lower part of the beam presents a depth a, and is assumed to be subjected to Mode I loading (i.e. the crack is normal to the longitudinal axis of the beam). Unidirectional fibers are discretely distributed across the crack and oriented along the longitudinal axis of the beam. The position of the *i*-th fiber ($i = 1,...,n$) is described by the distance c_i with respect to the bottom of the beam cross-section. Further, the relative crack depth $\xi = a / b$ and the normalized coordinate $\zeta_i = c_i / b$ are defined.

The matrix is assumed to present a linear elastic constitutive law, whereas the fibers are assumed to behave as rigid-perfectly plastic bridging elements which connect together the two surfaces of the crack. Hence, the rigid-perfectly plastic bridging law of the generic *i*-th fiber is characterized by an ultimate force $F_{P,\,i}$ (and $-F_{P,\,i}$ in compression), whichever of them exhibits the minimum absolute value [10, 11].

During the general loading process, brittle catastrophic fracture or compressive crushing of the matrix are disregarded. Further, no edge crack is assumed to develop in the upper part of the beam. Stable fatigue propagation of the initial crack due to cyclic loading is beyond the scope of the present investigation.

Since the problem being examined is statically indeterminate, the unknown fiber reactions F_i (positive if the fiber is under tensile loading) on the matrix can be deduced from n kinematic conditions related to the crack opening displacements w_i at the different fiber levels [10]. If $\left|F_i\right|$ is equal to $F_{P,\,i}$, the force of the *i*-th fiber becomes known, and the crack opening displacements are hereafter shown to depend on such a value. Since the matrix is assumed to behave in a linear elastic manner, the crack opening displacement w_i at the *i*-th fiber level is computed through the superposition principle

$$\mathbf{w} = \boldsymbol{\lambda}_{\mathbf{N}} N + \boldsymbol{\lambda}_{\mathbf{M}} M + \boldsymbol{\lambda}\,\mathbf{F} \tag{1}$$

where $\mathbf{w} = \{w_1,..., w_n\}^T$ is the vector of the crack opening displacements at the different fiber levels, and $\mathbf{F} = \{F_1,..., F_n\}^T$ is the vector of the crack bridging forces. Further,

$\boldsymbol{\lambda}_{\mathbf{N}} = \{\lambda_{1N},..., w_{nN}\}^T$ is the vector of the compliances related to the axial force *N*, $\boldsymbol{\lambda}_{\mathbf{M}} = \{\lambda_{1M},..., w_{nM}\}^T$ is the vector of the compliances related to the bending moment *M*, whereas $\boldsymbol{\lambda}$ is a symmetric square matrix of order *n*, whose generic element *ij* represents the compliance λ_{ij} related to the *i*-th crack opening displacement and the *j*-th fiber force (see Ref. [10] and the analytical expressions of SIFs in Ref. [14], pp. 52, 55, 71).

The incremental form of the governing Eq. 1 is (summation rule for repeated indices holds)

$$\dot{w}_i = \lambda_{iN}\,\dot{N} + \lambda_{iM}\,\dot{M} - \lambda_{ij}\,\dot{F}_j \qquad \text{with } \; i = 1,...,n \tag{2}$$

where dot symbol indicates time derivatives, being time the ordering variable, with $F_i = \int \dot{F}_i\,dt$ and $w_i = \int \dot{w}_i\,dt$. If the general *i*-th fiber is in the elastic domain, the corresponding increment of crack opening displacement $\dot{w}_i$ is null, namely if $|F_i| < F_{P,i} \Rightarrow \dot{w}_i = 0$. On the other hand, if the general *i*-th fiber is yielded ($|F_i| = F_{P,i}$), the following two alternatives are possible: $\dot{F}_i = 0 \Rightarrow F_i\,\dot{w}_i > 0$ or $F_i\,\dot{F}_i < 0 \Rightarrow \dot{w}_i = 0$ (plastic-to-elastic return). In other words, we have :

$$F_i\,\dot{w}_i > 0 \;\text{ if }\; |F_i| = F_{P,i} \;\text{ and }\; \dot{F}_i = 0\,; \qquad \dot{w}_i = 0 \;\text{ otherwise} \tag{3}$$

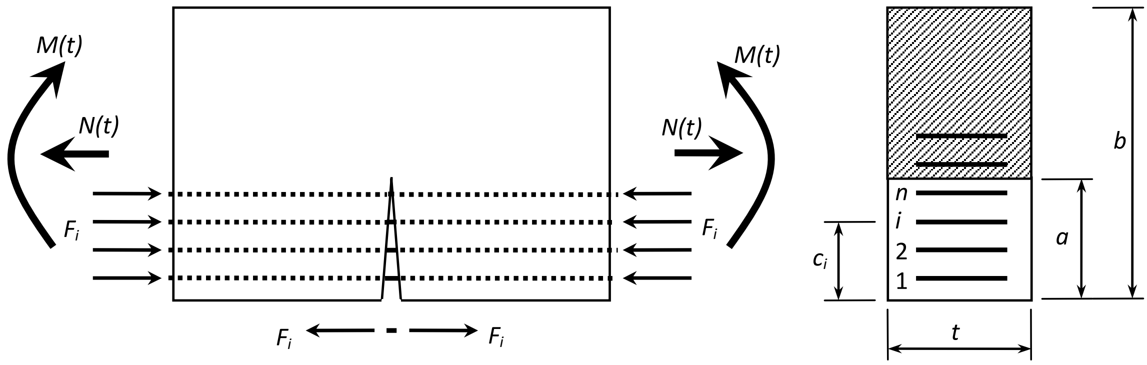

Figure 1. Schematic of the model.

3 Shakedown and the Melan's theorem in plasticity

Let us consider a body made of an elastic-perfectly plastic material. Strain is additively decomposed into elastic and plastic parts:

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} \tag{4}$$

The plastic strain is defined by a convex yield condition

$$\varphi(\sigma_{ij}) \le 0 \tag{5}$$

and the associated flow rule

$$\dot{\varepsilon}_{ij}^{p} = \dot{\alpha}\,\frac{\partial \varphi}{\partial \sigma_{ij}} \tag{6}$$

where $\dot{\alpha}$ indicates a non-negative scalar plastic multiplier ($\dot{\alpha} > 0$ if $\varphi = 0$ and $\dot{\varphi} = 0$). The Drucker's stability postulate holds [15]:

$$\left(\sigma_{ij}-\sigma_{ij}^{*}\right)\dot{\varepsilon}_{ij}^{p}\geq 0 \quad \forall \sigma_{ij}^{*},\sigma_{ij} \;\text{ such that }\; \varphi\left(\sigma_{ij}^{*}\right)\leq 0 \;\text{ and }\; \varphi\left(\sigma_{ij}\right)=0 \tag{7}$$

The elastic-plastic body under consideration is submitted to cyclic external loading with period *T*, such that an initial transient stage (leading to some possible mean values of load components) is followed by a cyclic stage. At a certain instant, the material attains (possibly asymptotically) a steady state, where the stress becomes a periodic function with period equal to that of the external loading, that is, for a sufficiently high value of t (possibly for $t \to \infty$), we get $\sigma_{ij}(t)=\sigma_{ij}(t+T)$. If plastic strain does not occur in the steady state but it is limited to a initial transient stage, shakedown (or adaptation) occurs.

Shakedown conditions can be ruled out if one considers the linear elastic response σ_{ij}^{e} of the body under the external loading, namely the stress state, satisfying the equations of elastic equilibrium, that would develop in the body if its behaviour were perfectly elastic. Such a time-varying stress state $\sigma_{ij}^{e}(\mathbf{x},t)$, function of the material point position $\mathbf{x}$, is linked by a one-to-one relation to the load path. If a body (submitted to a given periodic load path) shakes down, clearly there must exist (*necessary condition*) a time-independent self-balanced stress $\bar{\sigma}_{ij}^{p}(\mathbf{x})$ ($\bar{\sigma}_{ij,i}^{p}=0$ in V and $\bar{\sigma}_{ij}^{p}n_i=0$ on S_F) such that

$$\varphi\left(\sigma_{ij}^{e}(\mathbf{x},t)+\bar{\sigma}_{ij}^{p}(\mathbf{x})\right)\leq 0 \qquad \forall \mathbf{x},t \tag{8}$$

The Melan's theorem [13] supplies a *sufficient condition* for shakedown, and its statement is as follows: for given load path, an elastic-perfectly plastic body will shake down if and only if there exists a time-independent self-balanced (residual) stress $\rho_{ij}(\mathbf{x})$ ($\rho_{ij,i}=0$ in V and $\rho_{ij}n_i=0$ on S_F) that nowhere violates the yield criterion when superimposed onto the elastic stress in equilibrium with the given load path, that is (note the strict inequality):

$$\varphi\left(\sigma_{ij}^{e}(\mathbf{x},t)+\rho_{ij}(\mathbf{x})\right)< 0 \qquad \forall \mathbf{x},t \tag{9}$$

An evident advantage of the Melan's theorem is that the actual time-dependent elastic-plastic stress in the body does not have to be determined and, hence, no incremental analysis is required to assess shakedown conditions. Instead, the elastic solution $\sigma_{ij}^{e}(\mathbf{x},t)$ is superimposed on a self-balanced stress distribution (which may be different from the actual one caused by the given load path) so that the resulting stress state is admissible with respect to yielding.

Usually the external load path is not known *a priori* so that, typically, a family of load paths is considered by defining a load domain given by the max/min values of each single load component. In more details, let the vector $\mathbf{P}(t)$ collect the independent load components $\alpha_h(t)P_{h,\max}$ ($h=1,...,p$) with $\alpha_h(t)$ time varying between $\alpha_{h,\min}$ and $\alpha_{h,\max}=\mu_h$. Therefore, the load domain is bounded by hyperplanes, and the shakedown condition of Eq. 9 has to be verified at a finite number of points corresponding to the vertexes, intersections of the hyperplanes. If a proportional variation of the ranges of load components is assumed, $\mu_1=...=\mu_p=\mu$ and, hence, the load domain varies in a homothetic manner, defined by the single load parameter μ. In such a case, the maximum value of the load parameter μ defines the shakedown limit.

Extensions of the classical shakedown theory (concerned with elastic-perfectly plastic materials in small displacements and strains) to more general material models (such as to include non-linear hardening, rate-dependence, damage, non associative plasticity) and to large displacements have been formulated (e.g. see Ref. [16]). Investigations on shakedown conditions in elastic contact problems with Coulomb friction can be found in Ref. [17].

4. Shakedown limit in the crack bridging model

Shakedown theory related to plasticity material model can be extended to the case of the present crack bridging model on the basis of the following similarities. As a matter of fact, the stress state obtained from a linear elastic analysis can be regarded as the fiber forces due to zero crack opening displacements, that is, the forces which are proportional to the applied loads (axial force *N(t)* and bending moment *M(t)*). Hence such a force vector $\mathbf{F}^{(0)}(t)$ can be obtained by equating the right-hand member of Eq. 1 to zero:

$$\mathbf{F}^{(0)}(t) = \boldsymbol{\lambda}^{-1}\big(\boldsymbol{\lambda}_N\ N(t) + \boldsymbol{\lambda}_M\ M(t)\big) \tag{10}$$

Then, the time-independent residual stress corresponds to the fiber forces $\mathbf{F}^{(\tilde{w})}$ due to non-zero crack opening displacements $\tilde{\mathbf{w}}$, namely according to Eq. 1:

$$\mathbf{F}^{(\tilde{w})} = -\boldsymbol{\lambda}^{-1}\,\tilde{\mathbf{w}} \tag{11}$$

The sufficient condition of the Melan's shakedown theorem can hence be written as follows:

$$\left|\mathbf{F}^{(0)}(t) + \mathbf{F}^{(\tilde{w})}\right| < \mathbf{F}_p \quad \forall t \tag{12}$$

Considering the case of the rectangular load domain defined by $N(t) = \alpha_N(t) N_{\max}$ ($\alpha_{N,\min} \le \alpha_N(t) \le \mu_N$) and $M(t) = \alpha_M(t) M_{\max}$ ($\alpha_{M,\min} \le \alpha_M(t) \le \mu_M$), the condition of Eq. Eq. 12 has to be verified at the 4 vertexes (e.g. see the box path in Fig. 3c below). If one assumes a homothetic variation of the load domain, a single load parameter $\mu_N = \mu_M = \mu$ is considered, and the shakedown limit is obtained from the following optimization procedure:

$$\mu_{SD} = \max_{\mu,\{\tilde{\mathrm{w}}\}\ge 0}(\mu) \tag{13}$$

such that

$$\begin{aligned}
&\left|\boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_N\ \alpha_{N,\min} N_{\max} + \boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_M\ \alpha_{M,\min} M_{\max} - \boldsymbol{\lambda}^{-1}\tilde{\mathbf{w}}\right| < \mathbf{F}_p \\
&\left|\boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_N\ \alpha_{N,\min} N_{\max} + \boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_M\ \mu M_{\max} - \boldsymbol{\lambda}^{-1}\tilde{\mathbf{w}}\right| < \mathbf{F}_p \\
&\left|\boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_N\ \mu N_{\max} + \boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_M\ \mu M_{\max} - \boldsymbol{\lambda}^{-1}\tilde{\mathbf{w}}\right| < \mathbf{F}_p \\
&\left|\boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_N\ \mu N_{\max} + \boldsymbol{\lambda}^{-1}\boldsymbol{\lambda}_M\ \alpha_{M,\min} M_{\max} - \boldsymbol{\lambda}^{-1}\tilde{\mathbf{w}}\right| < \mathbf{F}_p
\end{aligned} \tag{14}$$

5. Illustrative example

For illustrative purposes, let us consider a fiber-reinforced concrete edge-cracked beam with height *b* = 0.3 m, thickness *t* = 0.2 m, relative crack depth $\xi = 0.2$, submitted to general combinations of periodic axial force and bending moment within a rectangular domain defined by their min/max values. Let us assume that $\alpha_{N,\min} = 0.1 N_p / N_{\max}$ and $\alpha_{M,\min} = 0.1 M_p / M_{\max}$, where N_p and M_p are the axial force and the bending moment, respectively, of first yielding in the most highly stressed fiber. The concrete Young modulus *E* is assumed to be equal to 30 GPa. Further, the concrete compressive strength and fracture toughness are assumed to be as high as to avoid crushing and brittle fracture of the matrix, respectively. Equally-spaced fibers characterized by an ultimate force F_p = 5266 N and a spacing Δc = 2.2 mm are considered (case representative of long steel fibers of 0.5 mm diameter, 5% volume fraction, 300 MPa yield stress), so that 27 fibers are intersected by the crack. By means the optimization procedure in Eqs 13 and 14, the shakedown limit can be obtained as the ratio $M_{\max} / N_{\max}$ is made to vary. This leads to the normalized Bree-like diagram in Fig. 2, where the elastic domain is also sketched ($N_p = 715113$ N and $M_p = 358083$ Nm).

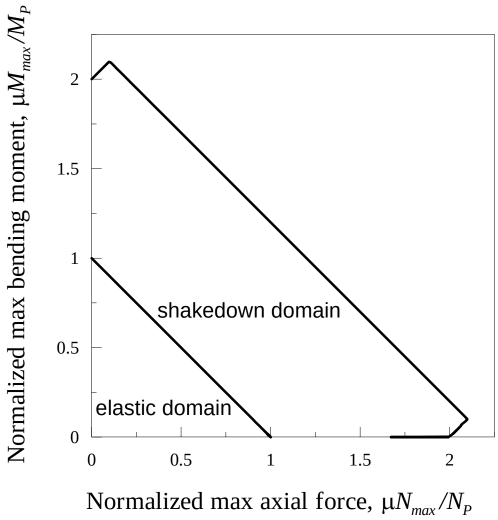

Figure 2. Bree-like diagram showing the elastic domain and the shakedown domain

In order to show the validity of the optimization procedure for determining shakedown limit (Eqs 13 and 14), two different periodic load paths (box path, 90° out-of-phase) are analysed (see Fig. 3, where the two histories of the loads are presented for the first 5 cycles) by means

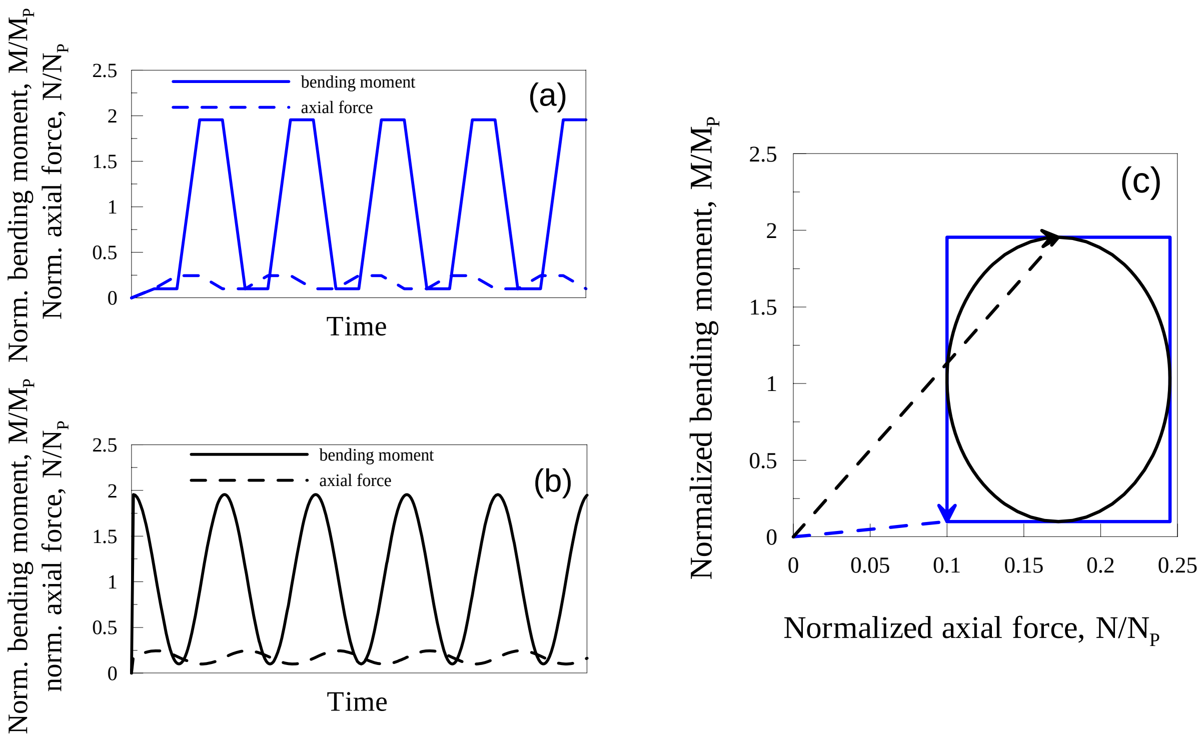

Figure 3. Load time-histories in the case of $M_{\max} / N_{\max}$ = 0.125 and μ_{SD} for (a) box path, (b) 90° out-of-phase path, and (c) corresponding load paths.

of the incremental procedure presented in Section 2 (see Eqs 2 and 3). The ratio $M_{\max} / N_{\max}$ is chosen to be equal to 0.4, and the load parameter μ is taken to be equal to μ_{SD} and to $1.05\mu_{SD}$.

In Fig. 4, the bridging force against crack opening displacement curves of the 1st (bottom) and 27th (top) fibers for $\mu = \mu_{SD}$ and $\mu = 1.05\mu_{SD}$ are plotted. It is evident that all the fibers shake down (the 2nd fiber to the 26th fiber are in intermediate conditions with respect to the 1st fiber and the 27th fiber) when $\mu = \mu_{SD}$ regardless of the characteristics of the load paths being considered. On the other hand, for $\mu = 1.05\mu_{SD}$, alternating plasticity with energy dissipation in hysteretic loops at the fiber levels takes place in the case of box path but not in the case of 90° out-of-phase path. This is a consequence of the fact that the optimization procedure in Eqs 13 and 14 ensures that, for *any* load path *within* the rectangular domain defined by μ_{SD} and a given $M_{\max} / N_{\max}$ ratio, the composite beam will shake down, but nothing is said about the shakedown condition for load paths *outside* the load domain. In such cases shakedown conditions will depend on a finite number of $N - M$ couples along the load path which describe a convex domain enclosing the load path.

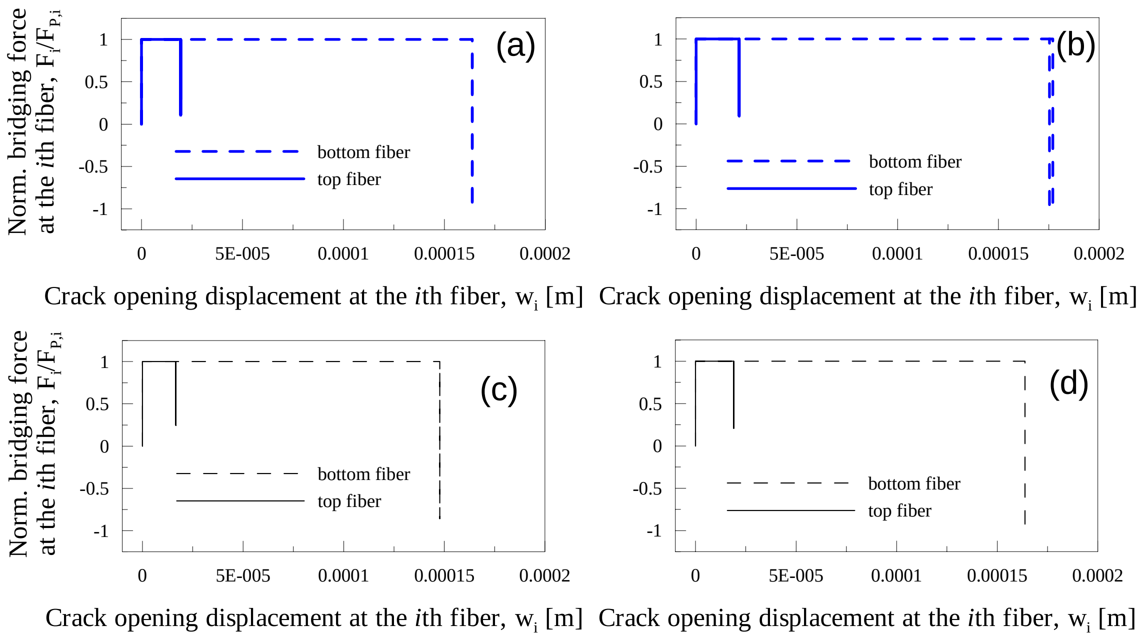

Figure 4. Bridging force vs crack opening displacement for: (a) box path and $\mu = \mu_{SD}$; (b) box path and $\mu = 1.05\mu_{SD}$; (c) 90° out-of-phase path and $\mu = \mu_{SD}$; (d) 90° out-of-phase path and $\mu = 1.05\mu_{SD}$.

6. Conclusions

In this paper, a bridging crack model for a fiber-reinforced brittle-matrix composite beam under oscillatory axial load and bending moment is presented. A simple rigid-perfectly plastic bridging law due to fibers is considered. By drawing a parallel with the well known problem of shakedown in elastic-perfectly plastic monolithic bodies, it is shown that the classical Melan’s theorem of limit analysis can successfully be applied to the present model, where the crack opening displacements at the fiber levels play the role of the plastic strains in monolithic bodies. For illustrative purposes, the results of the optimization procedure based on the Melan’s theorem are verified for a fiber-reinforced concrete beam by performing a step-by-step incremental procedure.

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