3D FINITE LAYER METHOD FOR THE ANALYSIS OF PIEZOELECTRIC COMPOSITE PLATES

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Abstract
Piezoelectric composite plates constitute an important element of smart structures. These structures are often exposed to high temperature environments. When the thermal stresses increase to a certain level, the structure may lose its elastic stability, and thermal buckling occurs. Most of the published results are approximate solutions obtained by means of simplified two-dimensional models. Three dimensional solutions are desirable in order to investigate performance of such structures more accurately, particularly for thick plates. A 3D finite layer method is used for the analysis of piezoelectric composite plates; in particular for the thermal buckling of antisymmetric angle-ply piezoelectric composite plates. The effects of material properties and structural geometry are investigated and the results are compared with available solutions

1 Introduction
Piezoelectric composite plates composed of piezoelectric layers and fibre-reinforced plastic layers constitute an important element of smart structures. These structures are often exposed to high temperature environments. As temperature rises, the thermal stresses in the structure build up. When the thermal stresses increase to a certain level, the structure may lose its elastic stability, and thermal buckling occurs. Several research works have been conducted to investigate the thermal buckling behaviours of smart composite plates. However, most of the published results are approximate solutions obtained by means of simplified two-dimensional models according to a variety of plate theories with varied complexities [1]. Therefore, whenever possible, three dimensional solutions are desirable in order to investigate performance of such structures more accurately. This is especially important for the relatively thick plates. The obtained results will provide important information, which cannot be extracted from two-dimensional analysis, and the solutions will also be useful for verification of other analysis tools such as a new plate theory and related numerical methods. In this context, Kapuria and Achary [2] have recently published the exact 3D solution for thermal buckling of piezoelectric cross-ply laminates using state space approach, and important findings are presented. However, to the authors' knowledge, the 3D thermal buckling analysis of piezoelectric angle-ply laminates has not been reported.
For a simply supported rectangular plate, the finite layer method is the most efficient numerical method in the 3D analysis. In this method, each lamina in a composite plate is modeled by one or more finite layers. Within each finite layer, the trigonometric functions are used for the inplane interpolations of displacements, whereas the polynomials are employed

for the interpolations in the thickness direction. Thus, the three dimensional analysis is transformed into a series of one dimensional analyses by virtue of the orthogonal properties of the trigonometric functions. Consequently, the results are more accurate than any analysis based on the plate theory since the solutions are true 3D solutions. On the other hand, the analysis is more efficient than 2D finite element analysis because the 3D analysis has been reduced to one-dimensional. Recently, the present authors have conducted the finite layer thermal buckling analysis of piezoelectric cross-ply laminates, yielding satisfactory accuracy and efficiency [3].

In the present study, the finite layer method is extended to the 3D thermal buckling analysis of simply supported rectangular piezoelectric antisymmetric angle-ply laminates. The laminate may also include some symmetrical cross-plies. The present analysis takes into account the full coupling between the thermal, electrical and mechanical fields, whereas the material properties are assumed to be independent of both the temperature and the electric field. In addition, the pre-buckling state of the plate is assumed to be steady. Therefore, temperature distribution in the plate is independently determined based on the equation of heat conduction; and the associated thermal stresses are computed accordingly. Then, the geometrical stiffness matrix is formed in the same way as in the elastic 3D buckling analysis. The critical temperature rise and the buckling mode can be obtained by solving related matrix equations.

Numerical results are presented to verify the proposed method. The effects of material properties and structural geometry are investigated.

2 Coupled thermal, electrical, mechanical fields

For the materials with fully coupled thermal, electrical and mechanical fields, the constitutive equations can be given in the tensor notion as [1-4]:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} - e_{ijk}\mathrm{E}_k - \beta_{ij}\theta \tag{1}$$

$$D_i = e_{ijk}\varepsilon_{jk} + \eta_{ij}\mathrm{E}_j + p_i\theta \tag{2}$$

$$S = \beta_{ij}\varepsilon_{ij} + p_i E_i + \frac{\rho c_v}{T_\circ}\theta \tag{3}$$

where σ_{ij} and ε_{ij} are respectively the components of the stress tensor and strain tensor, D_i and E_i are respectively the components of electric displacement vector and electric field vector, S is the entropy, and θ is the temperature rise from the initial temperature $T_\circ$. The coefficients C_{ijkl}, e_{ijk} and η_{ij} denote respectively the elastic constants, the piezoelectric constants and the dielectric permittivity. The quantities β_{ij} and p_i refer respectively to the stress-temperature expansion coefficients and pyroelectric constants, whereas ρ and c_v respectively represent the mass density and the specific heat per unit mass at constant strain.

The strains components ε_{ij} have the following relationships with the displacement components u_i:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}u_{k,i}u_{k,j} \tag{4}$$

whereas the electric fields are related to the electrostatic potential ϕ as:

$$\mathrm{E}_i = -\phi_{,i} \tag{5}$$

In addition, the solution of a steady-state problem should satisfy the following variational equations to maintain the equilibriums of stresses, charges and heat, respectively:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij} dV = \int_A t_i \delta u_i \, dA \tag{6}$$

$$\int_V D_i \delta\phi_{,i} dV = \int_A D_n \delta\phi \, dA \tag{7}$$

$$\int_V \kappa_{ij}\theta_{,i}\delta\theta_{,j} dV = \int_A q_n \delta\theta \, dA \tag{8}$$

where t_i and q_n denote the applied surface traction and heat flux, respectively, V and A respectively stand for the volume and surface area of the considered domain, whereas κ_{ij} represent the thermal conductivity. For a steady-state problem, the temperature distribution can be determined from Eq. (8) independently, and Eq. (3) is not required in the corresponding solution procedure.

3 Pre-buckling solution

A uniform temperature rise T is considered in the present study in order to simplify comparison with other solutions. A set of uniform actuation potentials $\phi^\circ(z_k)$ may also be applied to the surfaces $z = z_k$ of piezoelectric layers. In addition, it is assumed that the plate remains flat without any out-of-plane bending and the inplane displacements remain zero as these loads increase proportionally until buckling occurs [2]. For the antisymmetrical angle-ply piezo-laminates, it can be verified that above loads only yield inplane stresses σ_x°, σ_y°, τ_{xy}°, strain ε_z° and electric field E_z°, which all remain constant within each layer. Thus, the initial stresses σ_x°, σ_y° and τ_{xy}° in each layer can be determined from the following constitutive equations:

$$\sigma_x^\circ = \overline{Q}_{13}\varepsilon_z^\circ - \bar{e}_{31}\mathrm{E}_z^\circ - \beta_{xx}T \tag{9}$$

$$\sigma_y^\circ = \overline{Q}_{23}\varepsilon_z^\circ - \bar{e}_{32}\mathrm{E}_z^\circ - \beta_{yy}T \tag{10}$$

$$\tau_{xy}^\circ = \overline{Q}_{63}\varepsilon_z^\circ - \bar{e}_{36}\mathrm{E}_z^\circ - \beta_{xy}T \tag{11}$$

$$\sigma_z^\circ = \overline{Q}_{33}\varepsilon_z^\circ - \bar{e}_{33}\mathrm{E}_z^\circ - \beta_{zz}T = 0 \tag{12}$$

$$D_z^\circ = \bar{e}_{33}\varepsilon_z^\circ + \eta_{zz}\mathrm{E}_z^\circ + p_3 T \tag{13}$$

where $\overline{Q}_{ij}$ are the elastic moduli, $\bar{e}_{ij}$ are the piezoelectric moduli, β_{xx}, β_{yy}, β_{xy} and β_{zz} are the stress-temperature expansion coefficients, η_{zz} is the dielectric permittivity, p_3 is the pyroelectric constant, the superscript $^\circ$ represents the pre-buckling state, and $\mathrm{E}_z^\circ = -\Delta\phi^\circ / t$ with t denoting the thickness of piezoelectric layer, and $\Delta\phi^\circ$ denoting the increment of ϕ° across the layer. For an actuator layer, the electrical field E_z° is given. Therefore, σ_x°, σ_y° and τ_{xy}° can be directly calculated by means of Eqs (9) to (11), after the value of ε_z° is obtained from Eq. (12). For a sensor layer, D_z° is equal to zero. Thus, ε_z° and E_z° can be obtained using Eqs (12) and (13). Then, σ_x°, σ_y° and τ_{xy}° are readily calculated from Eqs (9) to (11).

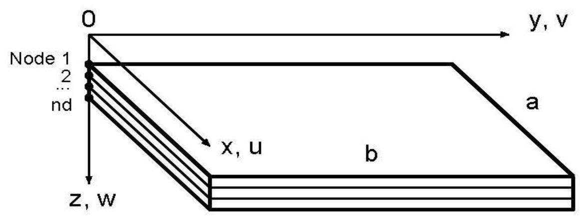

Figure 1. A Finite Layer

4 Finite layer method for buckling analysis

In the finite layer analysis of a piezoelectric composite plate, each material layer is modelled by one or more finite layers. Each finite layer has a number of equally spaced nodal planes, which are parallel to the plate middle plane $z = 0$, and labelled from the top to the bottom of the finite layer with i = 1, 2, … (Fig. 1). During buckling process of a simply supported rectangular antisymmetric angle-ply laminate, the increments of generalized displacements at any point within a finite layer are expressed in terms of the respective nodal values using the following interpolations:

$$u = \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{i=1}^{nd}\left(\hat{u}_{mni}N_i(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \tilde{u}_{mni}N_i(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)$$

$$v = \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{i=1}^{nd}\left(\hat{v}_{mni}N_i(z)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + \tilde{v}_{mni}N_i(z)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)$$

$$w = \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{i=1}^{nd}\left(\hat{w}_{mni}N_i(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \tilde{w}_{mni}N_i(z)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right) \tag{14}$$

$$\phi = \sum_{m=1}^{r}\sum_{n=1}^{s}\sum_{i=1}^{nd}\left(\hat{\phi}_{mni}N_i(z)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} + \tilde{\phi}_{mni}N_i(z)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\right)$$

where *u, v, w* denote incremental displacement components respectively in the x, y and z directions (Fig. 1), ϕ refers to incremental electrostatic potential, *r* and *s* are the numbers of series terms required in the analysis, *nd* is the selected number of nodal planes in a finite layer, $N_i(z)$ is the Lagrangian polynomial defined as:

$$N_i(z) = \prod_{j=1, j\neq i}^{nd} \frac{z - z_j}{z_i - z_j} \tag{15}$$

a and *b* are the side lengths of the plate in the x and y directions, respectively.

Using above shape functions and calculated initial inplane stresses, the generalized stiffness matrix and geometrical stiffness matrix of the finite layer can be formed by following standard procedures commonly used in the finite element and finite layer analysis [1, 3, 5]. Then, the critical temperature rise and buckling mode can be determined by solving related matrix equations. If material properties and initial inplane stresses remain constant in the inplane directions, the different pairs of series terms become uncoupled by virtue of the orthogonal properties of the trigonometric functions. Thus, the related matrix equations in the analysis can be formed and solved separately for each pair of series terms (*m*, *n*), so that the efficiency of the analysis is enhanced drastically

5 Numerical examples

5.1 Thermal buckling of square ($p/0^\circ/90^\circ/90^\circ/0^\circ/p$) piezoelectric laminates

A square piezoelectric laminate analyzed by Kapuria and Achary [2] is reanalyzed here to further validate the proposed method. The laminate with side length *a* and thickness *h* consists of a ($0^\circ/90^\circ/90^\circ/0^\circ$) sublaminate with four Graphite-Epoxy plies of 0.2*h* thickness each, and two continuous PZT-5A layers of thickness 0.1*h* each bonded to the upper and lower surfaces of the sublaminate. The material properties are given as:

Graphite-Epoxy: ($E_1, E_2, E_3, G_{12}, G_{23}, G_{31}$) = (181, 10.3, 10.3, 7.17, 2.87, 7.17) GPa
($\nu_{12}, \nu_{13}, \nu_{23}$) = (0.28, 0.28, 0.33); ($\alpha_1, \alpha_2, \alpha_3$) = (0.02, 22.5, 22.5)$\times 10^{-6}/^\circ K$

PZT-5A: ($E_1, E_2, E_3, G_{12}, G_{23}, G_{31}$) = (61.0, 61.0, 53.2, 22.6, 21.1, 21.1) GPa
($\nu_{12}, \nu_{13}, \nu_{23}$) = (0.35, 0.38, 0.38); ($\alpha_1, \alpha_2, \alpha_3$) = (1.5, 1.5, 2.0)$\times 10^{-6}/^\circ K$
($d_{31}, d_{32}, d_{33}, d_{15}, d_{24}$) = (-171, -171, 374, 584, 584)$\times 10^{-12} m/V$
($\eta_{11}, \eta_{22}, \eta_{33}$) = (1.53, 1.53, 1.5)$\times 10^{-8} F/m$; p_3 = 0.0007 $C/m^2 {}^\circ K$

All the four edges are simply supported with deflection and in-plane displacements being fixed on the middle plane along each edge. Two sets of electrical condition are considered, namely the closed-circuit (CC) condition with both the top and bottom surfaces of each piezoelectric layer being grounded and the open-circuit (OC) condition, where only the inner surface of each piezoelectric layer is grounded, while the outer surface remains free.

The resulting normalized critical temperature rise $\overline{T}_{cr} = \alpha_\circ T_{cr} a^2/h^2$ for $\alpha_\circ = 22.5\times 10^{-6} K^{-1}$ is given in Tab. 2. The exact three-dimensional solutions of Kapuria and Achary [2] are also listed in the same table for comparison. Results show that the proposed method yields excellent agreement with the exact 3D analysis.

a / h	**Closed-circuit condition**		**Open-circuit condition**	
	Present	**Kapuria**	**Present**	**Kapuria**
5	6.9954	-	3.8430	-
10	11.721	11.721	6.6242	6.6242
100	15.604	15.604	9.0142	9.0143

Table 1 Critical temperature rises $\overline{T}_{cr} = \alpha_\circ T_{cr} a^2/h^2$ of square ($p/0^\circ/90^\circ/90^\circ/0^\circ/p$) laminate

5.2 Thermal buckling of 10-layer square antisymmetric angle-ply laminate

The square antisymmetric angle-ply composite plate analyzed by Noor and Burton (1992) is selected here to verify the proposed method. The plate has side length *a*, thickness *h* and 10 equally thick layers alternatively oriented at θ and $-\theta$ degrees from the x-axis. The four edges are simply supported with deflection and normal inplane displacement fixed on middle plane. The material properties are assumed as: E_1 = 15 $E_\circ$, E_2 = E_3 = 1.0$E_\circ$, G_{12} = G_{13} = 0.5$E_\circ$, G_{23} = 0.3356$E_\circ$, ν_{12} = ν_{13} = 0.3, ν_{23} = 0.49, α_1 = 0.015$\alpha_\circ$, α_2 = α_3 = 1.0 $\alpha_\circ$, where $E_\circ$ and $\alpha_\circ$ are the normalization factors respectively for the elastic moduli and the coefficients of thermal expansion.

In all the examples of this study, each lamina is modelled by one finite layer with four nodal layers. The resulting normalized critical temperature rise $\overline{T}_{cr} = \alpha_\circ T_{cr}$ and corresponding buckling mode (m, n) are listed in Tab. 1 as the function of θ and thickness-to-length ratio *h/a*. It can be seen that the present results are in a close agreement with the exact 3D solutions of Noor and Burton [6].

A simply supported square piezoelectric laminate with length-to-thickness ratio *a/h*= 10 consists of a sublaminate with even number of equal thick Graphite-Epoxy layers alternatively oriented at θ° and $-\theta^\circ$ from the x-axis. Two continuous PZT-5A layers of thickness 0.1*h* each are bonded to the upper and lower surfaces of the sublaminate. All other parameters, conditions and analysis model are identical to the previous example. The resulting critical temperature rises $\bar{T}_{cr} = \alpha_\circ T_{cr} a^2/h^2$ for $\alpha_\circ = 22.5\times10^{-6} K^{-1}$ with buckling mode (1, 1) are shown in Fig. 3. It can be seen that $\bar{T}_{cr}$ reduces for the laminate with 2 sublaminate layers but rises for those with 4 and 8 sublaminate layers, as θ° increases from 0° to 45°.

h / a	Method	θ = 0 deg.	θ = 15 deg.	θ = 30 deg.	θ = 45 deg.
0.01	Present	0.7463×10^{-3}	0.1115×10^{-2}	0.1502×10^{-2}	0.1674×10^{-2}
	Noor	0.7463×10^{-3}	0.1115×10^{-2}	0.1502×10^{-2}	0.1674×10^{-2}
0.10	Present	0.5769×10^{-1}	0.7888×10^{-1}	0.1099	0.1193
	Noor	0.5782×10^{-1}	0.7904×10^{-1}	0.1100	0.1194
0.25	Present	0.1767	0.2073		
	Noor	0.1777	0.2087		
(m, n)		(1, 2)	(1, 2)	(1, 1)	(1, 1)

Table 2 $\bar{T}_{cr} = \alpha_\circ T_{cr}$ of 10-layer square antisymmetric angle-ply (θ /- θ …) laminates

5.3 Effect of angle θ° and number of layers

A simply supported square piezoelectric laminate with length-to-thickness ratio *a/h*= 10 consists of a sublaminate with even number of equal thick Graphite-Epoxy layers alternatively oriented at θ° and $-\theta^\circ$ from the x-axis. Two continuous PZT-5A layers of thickness 0.1*h* each are bonded to the upper and lower surfaces of the sublaminate. All other parameters, conditions and analysis model are identical to the previous example. The resulting critical temperature rises $\bar{T}_{cr} = \alpha_\circ T_{cr} a^2/h^2$ for $\alpha_\circ = 22.5\times10^{-6} K^{-1}$ with buckling mode (1, 1) are shown in Fig. 3. It can be seen that $\bar{T}_{cr}$ reduces for the laminate with 2 sublaminate layers but rises for those with 4 and 8 sublaminate layers, as θ° increases from 0° to 45°.

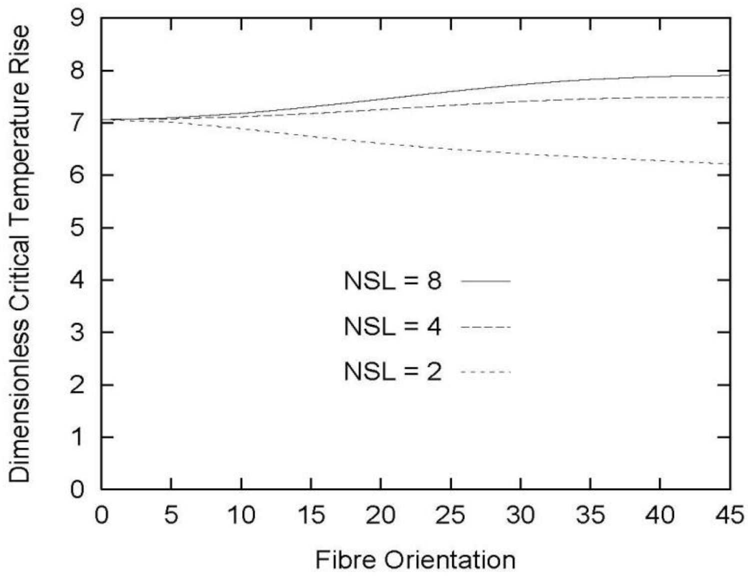

Figure 3. Effects of θ° on $\bar{T}_{cr}$ for square $(p/\theta^\circ/-\theta^\circ/\ldots/p)$ laminate of *a/h*=10 under open-circuit conditions (NSL = Number of Sublaminate Layers)

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