ON THE USE OF THREE DIMENSIONAL MATERIAL LAWS IN THE ANALYSES OF FUNCTIONALLY GRADED SHELLS

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Abstract
This work deals with the extension of the so-called 7-parameter shell model previously developed by Büchter et al. [Int. J. Numer. Meth. Engng. (1994) 37:2551-2568] to its usage in Functionally Graded Materials. Particularly, the material properties vary continuously through the thickness of the shell body according to a power law distribution that relates the volume fraction of the constituents. This structural model is implemented into the commercial FE code ABAQUS and tested by means of standard benchmark examples including geometrically nonlinear cases.

1. Introduction

Functionally Graded Materials (FGMs) are a particular case of composites in which the mechanical properties of the structural component vary continuously as a function of the position along one or more directions of the structure. They were originally conceived to be used as thermal barrier materials in aerospace applications as well as in other high temperature environments such as nuclear reactors and chemical plants, among others [1]. Furthermore, the possibility of adapting the internal material distribution to different potential applications makes them significantly attractive [2, 3].

In recent years, an important number of works have appeared in order to develop accurate structural models for the description of the structural response of FGM plates and shells. From the structural standpoint, most of this works deal with the adaptation of Classical Laminate (CLT), First-Order Shear Deformation (FOSDT) [4] and Higher Order Shear Deformation Theories [3, 5] for FE analyses of FGMs. These models are specifically based on the underlying structural shell models of Kirchhoff-Love (3-parameter) and Reissner-Mindlin (5-parameter). The mechanical hypothesis on which these structural models are based can be considered quite inappropriate due to the strong variation of the mechanical properties in thickness direction that these structures could present. In this regard, further expansions that allow more accurate kinematic description along the transverse normal direction of the shell were proposed in [6, 7].

This research deals with the extension of the 7-parameter shell model for the analysis of FGM [8, 9, 10]. This shell model was implemented into the commercial FEA package ABAQUS via user subroutine UEL, considering geometrically nonlinear cases.

2. Basic relations of the 7-parameter shell model

In this section, the basic features regarding the 7-parameter shell model and the corresponding FE formulation are briefly presented.

According to Figure 1, convective coordinates are introduced[1]. The covariant tangent vectors are obtained by partial derivatives of the position vectors with respect to the curvilinear coordinates:

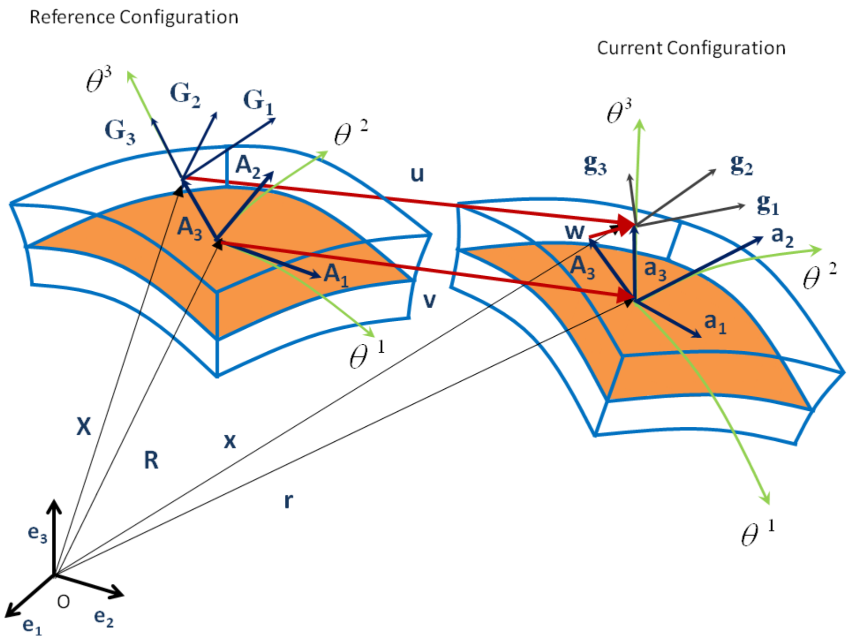

Figure 1. Kinematics of the 7-parameter shell model

$$\mathbf{G}_\alpha = \frac{\partial \mathbf{X}}{\partial \theta^\alpha} = \mathbf{X},_\alpha = \mathbf{A}_\alpha + \theta^3 \mathbf{A}_3,_\alpha \text{ with } \alpha = 1,2\,;\;\; \mathbf{G}_3 = \mathbf{A}_3 = \frac{h}{2}\frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|} \tag{1}$$

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} = \mathbf{x},_\alpha = \mathbf{a}_\alpha + \theta^3 \mathbf{a}_3,_\alpha \text{ with } \alpha = 1,2\,;\;\; \mathbf{g}_3 = \mathbf{a}_3 \tag{2}$$

$\mathbf{A}_\alpha$ and $\mathbf{a}_\alpha$ being

$$\mathbf{A}_\alpha = \frac{\partial \mathbf{R}}{\partial \theta^\alpha} = \mathbf{R},_\alpha \text{ and } \mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \theta^\alpha} = \mathbf{r},_\alpha \text{ with } \alpha = 1,2 \tag{3}$$

where $\mathbf{X}$ and $\mathbf{x}$ are the position vectors of an arbitrary point in the shell body, $\mathbf{R}$ and $\mathbf{r}$ denote the position vectors of the corresponding points on the shell midsurface, $\mathbf{A}_3$ and $\mathbf{a}_3$ (also denoted as $\mathbf{D}$ and $\mathbf{d}$ respectively) are the so-called director vector of the shell in both configurations, and

[1] In the following the capital letters are referred to the reference configuration of the shell, whereas the small letters are associated to the current configuration

h corresponds to the initial shell thickness. Thus, θ^1 and θ^2 denote the in-plane shell coordinates and θ^3 is the thickness coordinate. In the present shell model, the parametrization of the kinematic field of the continuum body is approximated by assuming a linear variation of the displacements in the thickness direction. Hence, the position vectors in the shell body in both configurations are expressed as:

$$\mathbf{X} = \mathbf{R} + \theta^3 \mathbf{A}_3 \quad \text{and} \quad \mathbf{x} = \mathbf{r} + \theta^3 \mathbf{a}_3 \tag{4}$$

By definition, the kinematic field is expressed as the difference of the position vectors of an arbitrary point of the continuum 3D body in the current and in the reference configurations: $\mathbf{u} = \mathbf{x} - \mathbf{X}$. Therefore,

$$\mathbf{u} = \mathbf{x} - \mathbf{X} = (\mathbf{r} - \mathbf{R}) + \theta^3 (\mathbf{a}_3 - \mathbf{A}_3) = \mathbf{v} + \theta^3 \mathbf{w} \tag{5}$$

The parametrization of the kinematic field is decomposed into: three translational degrees of freedom $\mathbf{v}$ (corresponding to the shell midsurface) and three difference degrees of freedom $\mathbf{w}$ denoting the relative displacement between the mid and the upper surfaces of the body. This difference vector is used for updating of the director vector of the shell along the deformation process (playing a similar role as the rotational degrees of freedom in the Kirchhoff-Love and in the Reissner-Mindlin formulations). It is worth mentioning that, as was deeply discussed in [8], this parametrization of the kinematic field of the shell incurs the so-called Poisson thickness locking.

2.1. Mixed FE formulation. Variational basis and locking discussion

The FE formulation of the present shell model is based on the three-field Hu-Washizu functional, where the displacement, incompatible strain and stress fields are the independent tensorial quantities. In the EAS Method, following the approach proposed in [9], the extra strain field $\tilde{\mathbf{E}}$ supplements the displacement compatible Green–Lagrange strain tensor $\mathbf{E}^u$. Hence, the strain field finally is expressed as: $\mathbf{E} = \mathbf{E}^u + \tilde{\mathbf{E}}$. In a Total Lagrangian formulation, the Hu-Washizu functional yields,

$$\tilde{\Pi}_{HW}(\mathbf{S}, \tilde{\mathbf{E}}, \mathbf{u}) = \int_{\Omega} \left[W_{3D}^{int}(\mathbf{E}^u + \tilde{\mathbf{E}}) - \tilde{\mathbf{S}} : \tilde{\mathbf{E}} \right] \mathrm{d}V + \Pi_{ext} \tag{6}$$

The first term of (6) describes the internal potential, W_{3D}^{int} is the internal strain energy stored and $\tilde{\mathbf{S}}$ denotes the Second Piola-Kirchhoff stress tensor. Finally, Π_{ext} corresponds to the external potential. The displacement compatible part of the strain tensor $\mathbf{E}^u$ can be expressed as:

$$\mathbf{E}^u = \frac{1}{2} \left[\mathbf{F}^T \cdot \mathbf{F} - \mathbf{G}_i \cdot \mathbf{G}_j \right] (\mathbf{A}^i \otimes \mathbf{A}^j) \tag{7}$$

where $\mathbf{F}$ denotes the Deformation Gradient tensor, the Green-Lagrange strain tensor being referred to the curvilinear basis on the shell midsurface. Hence the components of $\mathbf{E}^u$ in matrix notation can be split into the constant (α_{ij}) and the linear (β_{ij}) parts of the strain distribution across the thickness direction:

$$\mathbf{E}_{ij}^u = \alpha_{ij}^u + \theta^3 \beta_{ij}^u \tag{8}$$

By means of imposing the orthogonality condition between the independent stress field $\tilde{\mathbf{S}}$ and the enhanced strain field $\tilde{\mathbf{E}}$:

$$\int_{\Omega} \tilde{\mathbf{S}} : \tilde{\mathbf{E}} \, \mathrm{d}V = 0 \tag{9}$$

Therefore, the stress field is completely removed from the present formulation. Thus, the first variation of the functional (6) is obtained via the directional derivative concept [11]. The resulting nonlinear set of equations is solved iteratively thorugh the corresponding linearization procedure [9], which is typically performed in nonlinear FE formulations. Moreover, once the discretization process of the kinematic and the enhanced strain fields is accomplished, it is possible to preserve the pure displacement FE formulation by eliminating the enhanced strains using a standard static condensation scheme.

Regarding the FE formulation based on this shell model, first-order elements with linear interpolation of the displacement field in-plane direction are now presented. The discretization procedure of the structure results in bi-dimensional FE meshes, where the nodal locations correspond to each of the four corners of the element of the midsurface. Within the standard isoparametric concept, both the geometry of the elements and displacements are interpolated using the same bilinear shape functions, denoted by $\mathbf{N}$, through the nodal coordinates $\mathbf{X}_h$ and the nodal displacements $\mathbf{d}_u$,

$$\mathbf{X} \approx \mathbf{X}_h = \mathbf{N} \cdot \mathbf{R} + \theta^3 \mathbf{N} \cdot \mathbf{D}; \quad \mathbf{X}_h = \sum_{k=1}^{N} N^k \mathbf{X}^k \quad \text{k=1,4} \tag{10}$$

$$\mathbf{u} \approx \mathbf{u}_h = \mathbf{N} \cdot \mathbf{v} + \theta^3 \mathbf{N} \cdot \mathbf{w} = \mathbf{N} \cdot \mathbf{d}_u; \quad \mathbf{u}_h = \sum_{k=1}^{N} N^k \mathbf{d}_u^k \quad \text{k=1,4} \tag{11}$$

where the subscript h indicates the finite element approximation, N refers to the number of element nodes (4 in case of linear elements). This elements will be denoted in the following as Three-dimensional 7-parameter element or TShell in abbreviated form.

It is important to remark that different locking pathologies appear in the case of standard low-order finite element approximations, which lead to unrealistic (over-stiffened) numerical predictions. In this regard, several numerical procedures have been developed during the last three decades in order to remove this deficiency, such as the Enhanced Assumed Strain (EAS) [12] and the Assumed Natural Strain (ANS) [13] Method. In the present FE formulation, a combination of both procedures is employed for this purpose in agreement with the the scheme proposed in [14]. Specifically, the EAS Method is employed for the treatment of the Membrane and Poisson Thickness locking, whereas the ANS Method is used to alleviate the Transverse Shear locking [13] and the Curvature Thickness locking [15] effects.

3. Functionally Graded shells. Constitutive law

In this work, for the case of Functionally Graded Materials, a hyperelastic and inhomogeneous material law in the thickness direction is considered. The linear relation between the Second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Green-Lagrange strain tensor $\mathbf{E}$ is assumed, implying a

Kirchhoff-Saint-Venant material type. FGMs possess a microscopically inhomogeneous character, which is typically made from two isotropic constituents. One of the most commonly employed cases considers a ceramic-metallic structure, see Figure 2.

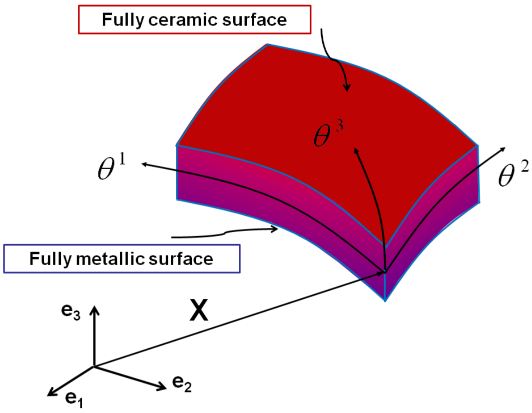

Figure 2. Functionally Graded Shell Structure

The material property gradation is taken into consideration by a function of the thickness coordinate (θ^3), see Figure 2. Consequently, the material properties at a certain point of the shell body **X** (in the reference configuration) is expressed by the weighted average of the moduli of the constituents, which is accomplished by the rule of mixtures based on the Voigt model [16]:

$$\iota(\theta^3) = \iota_m f_m + \iota_c f_c \tag{12}$$

where the subscripts m and c identify the metallic and the ceramic components respectively, f is the volume fraction of the corresponding phase, and ι denotes a generic material property such as the Young's Modulus and/or the Poisson ratio for isotropic materials, and some of the Young's Moduli and/or Poisson ratios in the case of orthotropic materials. According to [3] and [7], the volume fraction corresponding to the ceramic and the material constituents can be represented by the following functions of the thickness coordinate:

$$f_c = \left(\frac{\theta^3}{h} + \frac{1}{2}\right)^n \qquad f_m = 1 - f_c \tag{13}$$

where h denotes the thickness of the structure, and n is a volume fraction exponent which takes values greater than or equal to zero. Thus, the value of n equal to zero represents a fully ceramic structure, whereas when n tends to infinity a fully metallic shell is obtained, see Figure 3. In this work, the variation of the Young's modulus E according to the equation (13) is assumed, whereas the Poisson ratio ν is considered as a constant magnitude. Hence, the components of the elasticity tensor are functions of the thickness coordinate $C^{ijkl}(\theta^3)$:

$$\mathbb{C} = C^{ijkl}(\theta^3)\mathbf{G}_i \otimes \mathbf{G}_j \otimes \mathbf{G}_k \otimes \mathbf{G}_l \tag{14}$$

Therefore, the variation of the mechanical properties across the shell thickness is expressed as:

$$\begin{aligned} E(\theta^3) &= E_c f_c + E_m f_m \\ \nu(\theta^3) &= \nu \quad \text{is considered constant through the thickness} \end{aligned} \tag{15}$$

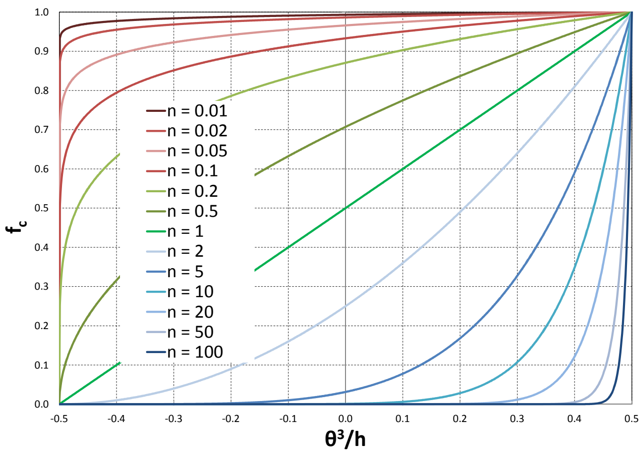

Figure 3. Variation of the volume fraction of ceramic material f_c through the thickness according to (13)

Hence,

$$C^{ijkl}(\theta^3) = C_c^{ijkl} f_c + C_m^{ijkl} f_m \tag{16}$$

with f_c and f_m given by equation (13) respectively.

4. Numerical application. Cantilever beam

This example concerns a cantilever beam subjected to uniform end forces, see Figure 4. A resulting mesh of 8 × 1 elements (Mesh 8) is used for its discretization, where the first index refers to 8 in-plane elements and the second one denotes 1 element over the thickness. In the case of FGMs, the material properties of the inhomogeneous beam vary continuously through the thickness direction according to equation (13). The mechanical properties of the two constituents and the mechanical load applied to the structures are shown in Figure 4. Figure 5

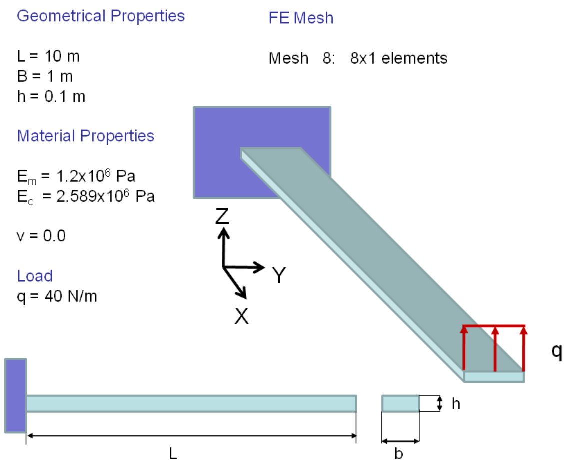

Figure 4. Functionally Graded cantilever beam under distributed end force

depicts the tip displacements of the cantilever beam in x and z directions vs. the external load applied for various volume fraction exponents n, which varies from the fully ceramic surface to the fully metallic surface. The Newton-Raphson Method exhibits a good ratio of convergence for all the cases. It is worth mentioning that, as was expected, the bending response of the FGMs are limited between the response of the fully metallic and the fully ceramic applications.

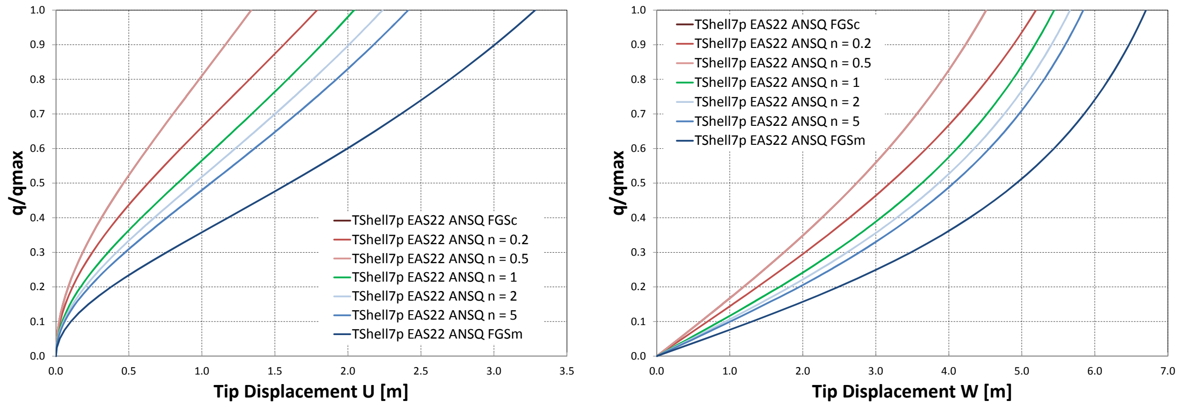

Figure 5. Tip cantilever beam subjected to end forces FGMs. Load deflection curves for the TShell7p element

5. Concluding remarks

This paper proposed the extension of the 7-parameter shell model to be applied to FGMs. This shell model allow the unmodified three-dimensional constitutive material law. Therefore, the transverse normal strain effects can be considered, and thus leading to a further insight into the three-dimensional character of the shell structures and keeping the bi-dimensional character of the resulting FE meshes. The numerical implementation of this extension has been successfully implemented into the ABAQUS code for geometrically linear and nonlinear applications including also the necessary numerical techniques to avoid undesirable locking effects. In order to illustrate this task, a classical benchmark problem is presented in this research.

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