A NOVEL RATIONAL DESIGN METHOD FOR LAMINATED COMPOSITE STRUCTURES EXHIBITING COMPLEX GEOMETRICALLY NONLINEAR BUCKLING BEHAVIOUR

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Abstract
This paper presents a novel FEM-based approach for fiber angle optimal design of laminated composite structures exhibiting complicated nonlinear buckling behavior, thus enabling design of lighter and more cost-effective structures. The approach accounts for the geometrically nonlinear behavior of the structure by utilizing path tracing response analysis up until the buckling point. The method simultaneously includes loss of stability due to bifurcation and limiting behavior and thereby avoids problems related to mode or stability type switching during optimization. The optimization formulation is formulated as a mathematical programming problem and solved using gradient-based techniques.

1 Introduction
Polymeric resin fibre reinforced materials (FRP´s or composite materials) are being used increasingly for structural applications where properties such as high strength, high stiffness and low weight are determining design parameters. The driving force behind the development and application of these materials has been the demands posed by the aerospace industry, but the use of advanced composite materials is expanding rapidly to other industrial sectors including marine/off-shore, wind turbines (blades), automotive, train and civil engineering applications. Designing structures made out of composite material represents a challenging task, since both thicknesses, number of plies in the laminate and their relative orientation must be selected. The best use of the capabilities of the material can only be gained through a careful selection of the layup. This work focuses on optimal design of laminated composite shell structures with focus on the optimal fiber orientations within the laminate which is a complicated problem due to the many possible design combinations.

Stability is one of the most important objectives/constraints in structural optimization of shell structures, such as wind turbine blades. In stability analysis the buckling behaviour is often considered by linearized eigenvalue analysis alone, without any consideration to the type of buckling, generally resulting in overestimated buckling loads. Recent studies in [1] on buckling optimization of structures with geometrically nonlinear behaviour show that formulations based on linear buckling analysis may lead to unreliable design results. At present, only limit point buckling has been applied for nonlinear buckling optimization of laminated composite structures, see [1,2]. Thus, there is a lack of optimization procedures that handles bifurcation instability with nonlinear prebuckling effects together with optimization procedures that simultaneously handles bifurcation and limit point instability.

This paper focuses on the development of an integrated and reliable method for doing optimization of composite structures wrt. a general type of instability. Different types of buckling behaviour are defined and characterized by studying a well-known benchmark problem of a point loaded curved shell panel, first introduced by [3], and later used extensively in literature to investigate advances in finite elements for handling load and deflection reversals in nonlinear buckling problems. The structural response of the benchmark problem, as reported in many research papers through several decades, was lately discovered to be incorrect by [4,5]. However, the solution reported by [4,5] is found not to be without flaws. New features of the benchmark problem are revealed and includes discovery of an asymmetric buckling solution in the form of an unstable symmetric point of bifurcation.

The buckling benchmark problem will be the foundation for discussing and defining the challenges that may be encountered while optimizing geometrically nonlinear structures wrt. a general type of instability. The proposed method includes the nonlinear prebuckling effects by using geometrically nonlinear path tracing analysis by the arclength method. The nonlinear analysis is stopped when a buckling point is encountered and the buckling load is approximated at a precritical load by an eigenvalue analysis on the deformed configuration. The optimization formulation is formulated as a mathematical programming problem and solved using gradient-based methods. Design sensitivities of the critical load factor are obtained semi-analytically by the direct differentiation approach on the approximate eigenvalue problem described by discretized finite element matrix equations.

The proposed optimization procedure is benchmarked against a formulation based on linear buckling analysis of a shell buckling problem which helps to clarify the importance of including nonlinear prebuckling effects in structural design optimization wrt. stability.

2 Nonlinear buckling analysis of composite structures

Structural stability/buckling is estimated in terms of geometrically nonlinear analyses and applies for both bifurcation and limit point instability, depending on what to appear on the equilibrium path. The proposed procedure for nonlinear buckling analysis is schematically shown in Fig. 1 and consists of the steps stated in Algorithm 1. During a geometrically nonlinear analysis the fundamental stability point is detected if it exists. Two instability situations are depicted in Fig. 1, an unstable bifurcation point and a load limit point.

We consider geometrically nonlinear behaviour of structures made of linear elastic materials. We adopt the Total Lagrangian approach, i.e. displacements refer to the initial configuration, for the description of geometric nonlinearity. An incremental formulation is more suitable for nonlinear problems and it is assumed that equilibrium at load step n is known and it is desired at load step n+1. Furthermore, it is assumed that the current load is independent on deformation. The incremental equilibrium equation is given as

$$\begin{aligned} &\mathbf{K_T}(\mathbf{D}^n,\gamma^n)\,\delta\mathbf{D} = \mathbf{R}^{n+1} - \mathbf{F}^n \\ \text{where}\quad &\mathbf{K_T}(\mathbf{D}^n,\gamma^n) = \mathbf{K_0} + \mathbf{K_L}(\mathbf{D}^n,\gamma^n) + \mathbf{K_\sigma}(\mathbf{D}^n,\gamma^n) \\ \text{i.e.}\quad &\mathbf{K}_\mathbf{T}^n = \mathbf{K_0} + \mathbf{K}_\mathbf{L}^n + \mathbf{K}_\sigma^n \end{aligned} \tag{1}$$

Here $\delta\mathbf{D}$ is the incremental global displacement vector, $\mathbf{F}^n$ global internal force vector, and $\mathbf{R}^{n+1}$ global applied load vector. The global tangent stiffness $\mathbf{K}_\mathbf{T}^n$ consists of the global initial stiffness $\mathbf{K_0}$, the global stress stiffness $\mathbf{K}_\sigma^n$, and the global displacement stiffness $\mathbf{K}_\mathbf{L}^n$. The applied load vector $\mathbf{R}^n$ is controlled by the stage control parameter (load factor) γ^n according to an applied reference load vector $\mathbf{R}$, i.e. $\mathbf{R}^n = \gamma^n\mathbf{R}$.

Algorithm 1: Pseudo code for the nonlinear buckling analysis	
1:	Geometrically nonlinear (GNL) analysis by arclength method
2:	Monitor and detect stability point during GNL analysis
3:	Re-set all state variables to configuration at load step just before stability point – a precritical point
4:	Perform eigenbuckling analysis on deformed configuration at load step before stability point

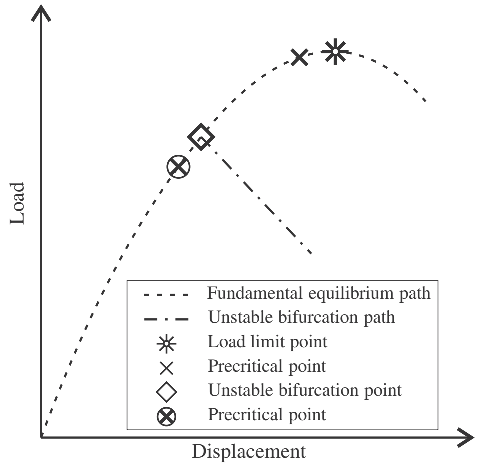

Figure 1. Detection of stability point in step 2 and chosen precritical equilibrium point for the nonlinear buckling problem in case of bifurcation and limit point instability.

The incremental equilibrium equation (1) is solved by the spherical arclength method after [6]. At a critical point the tangent operator is singular

$$\mathbf{K_T}(\mathbf{D}^c, \gamma^c)\boldsymbol{\phi}_j = \mathbf{0} \tag{2}$$

where the superscript *c* denotes the critical point and $\boldsymbol{\phi}_j$ the buckling mode. To avoid a direct singularity check of the tangent stiffness, it is easier to utilize tangent information at some converged load step *n* and extrapolate it to the critical point. The stress stiffness part of the tangent stiffness at the critical point is approximated by extrapolating the nonlinear stress stiffness from the current configuration as a linear function of the load factor γ, whereas it is assumed that the initial stiffness and displacement stiffness do not change with additional loading. This holds if the additional displacements are small. The nonlinear buckling problem can now be expressed as a generalized eigenvalue problem for the equilibrium configuration at load step *n* as

$$\begin{aligned} &\left(\mathbf{K_0} + \mathbf{K}_\mathbf{L}^n\right)\boldsymbol{\phi}_j = -\lambda_j \mathbf{K}_\sigma^n \boldsymbol{\phi}_j \\ &\text{with } \ \gamma_j^c = \lambda_j \gamma^n \end{aligned} \tag{3}$$

where the eigenvalues are assumed ordered by magnitude such that λ_1 is the lowest eigenvalue and $\boldsymbol{\phi}_1$ the corresponding eigenvector. If $\lambda_1 < 1$ the first critical point has been passed and in contrary $\lambda_1 > 1$ the critical point is upcoming. The closer the current load step gets to the critical point, the better the approximation becomes, and it converges to the exact result in the limit of the critical load.

3 Optimization formulation of the nonlinear buckling problem

To accomplish gradient-based optimization of the nonlinear buckling load factors, the nonlinear buckling load factor sensitivities are needed. Considering simple eigenvalues of conservative load systems, the eigenvalue sensitivity wrt. any design variable a_i, $i = 1, \dots, I$ can be derived as

$$\frac{d\lambda_j}{da_i} = \boldsymbol{\phi}_j^T \left(\frac{d\mathbf{K_0}}{da_i} + \frac{d\mathbf{K}_\mathbf{L}^n}{da_i} + \lambda_j \frac{d\mathbf{K}_\sigma^n}{da_i} \right) \boldsymbol{\phi}_j \tag{4}$$

The global matrix derivatives are determined semi-analytically utilizing central difference approximations on element level and assembled to global matrix derivatives.

The mathematical programming problem for maximizing the lowest critical load is a max-min problem. In order to avoid problems related to differentiability and fluctuations during the optimization process the bound formulation is utilized. The optimization formulation in the case of laminate optimization, for a max-min problem with the use of the bound formulation, is formulated as follows

$$\begin{aligned} \text{Objective :} \quad & \max_{\boldsymbol{a},\beta} \ \beta \\ \text{Subject to :} \quad & \gamma_j^c \geq \beta, \quad j = 1, \dots, N_\lambda \\ & \left(\mathbf{K_0} + \mathbf{K}_\mathbf{L}^n + \lambda_j^n \mathbf{K}_\sigma^n \right) \boldsymbol{\phi}_j^n = \mathbf{0} \\ & \gamma_j^c = \lambda_j^n \gamma^n \\ & \underline{a_i} \leq a_i \leq \overline{a_i}, \quad i = 1, \dots, I \end{aligned} \tag{5}$$

where a_i denote the laminate design variables in terms of fiber angles. The mathematical programming problem is solved by the Method of Moving Asymptotes (MMA) by [7].

4 The cylindrical shell benchmark problem and solutions

The cylindrical shell example, see Fig. 2, first introduced by [3] is used to illustrate the complicated behaviour that may be encountered in shell buckling. Both the incorrect symmetric solution and the correct asymmetric solution to the benchmark problem are presented in order to clarify the complicated behavior that may be encountered in shell buckling for even an immediate simple well-known example. The study of the example will therefore pinpoint some of the challenges in optimizing geometrically nonlinear structures wrt. stability. The numerical results are obtained by simulations based on an in-house FE and optimization code called the MUltidisciplinay Synthesis Tool (MUST) and the commercial FE program ANSYS.

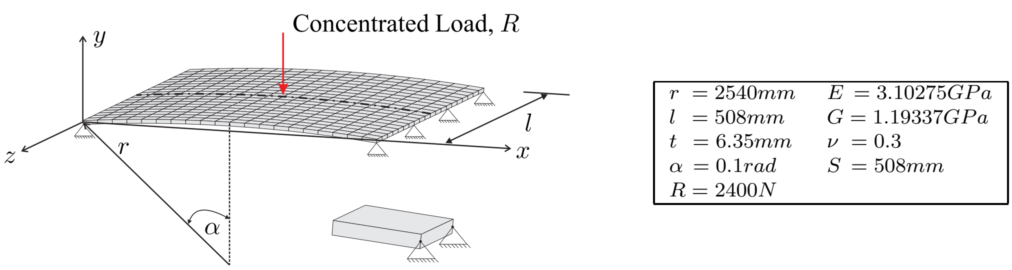

Figure 2. Geometry, loads, boundary conditions, and material properties for the cylindrical shell example. The hinged support is related to the mid surface of the shell, which is realized by multi point constraints between the top and bottom edge nodes. The shell is loaded by two point loads in the negative *y*-direction, at the top and bottom node in the centre of the segment. All dimensions refer to the mid surface, where the thickness is denoted by *t*. The shell centerline is marked on the figure and is represented by the bottom mesh grid points.

The isotropic shell panel, modelled by 400 equivalent single layer solid shell finite elements, is transversely loaded undergoing large deformations including buckling and post-buckling. The panel is supported by its two straight axial edges having a pinned fixture that cannot move, i.e. the mid-surface of the axial edges are restrained in displacements and rotations in *u, v, w, Rx, Ry* but free to rotate about the *z*-axis.

4.1 Symmetric solution

The symmetric solution, introduced by [3] and later reported by many authors, may be obtained by geometric nonlinear analysis upon the original perfect system. The stability limit is characterized by a load limit point, see Fig. 3. A path tracing algorithm as the arclength method after [6] is needed for this solution as both load and deflection reversals occur. Snap-through would occur at the load limit point in load control, and snap-down/snap-back at the deflection limit point in deflection control. Spanwise mode shapes along the shell centerline are symmetric about the centerline and loading point for the symmetric solution.

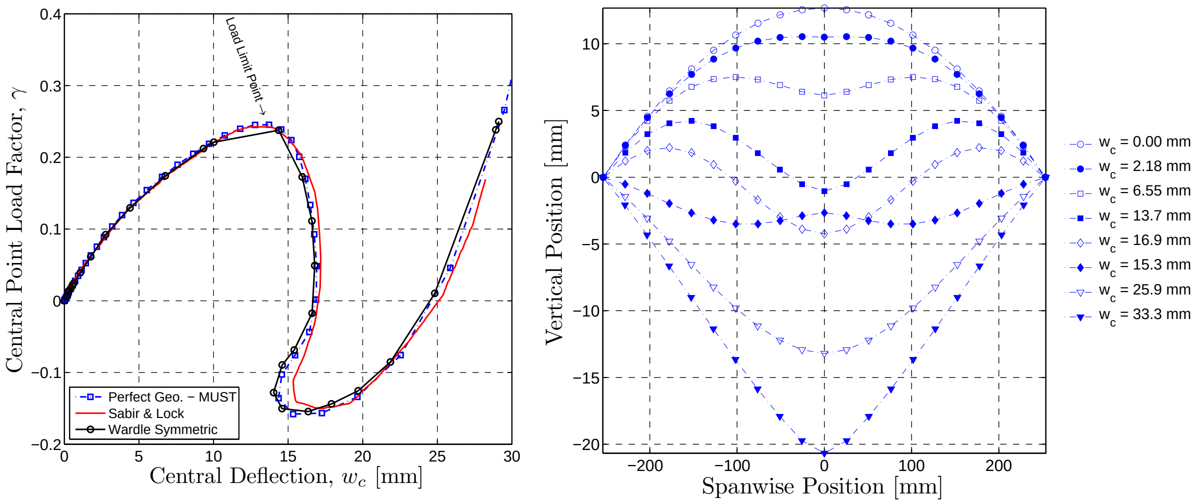

Figure 3. Left: Load-deflection response solutions of the perfect symmetric. Right: Central spanwise mode shapes at different values of center deflection, w_c, obtained by FEA on the perfect symmetric system.

4.2 Asymmetric solution

The symmetric solution of the problem makes the assumption that limit point buckling will occur and does not consider bifurcation and associated asymmetric buckling mode. Most analyses make this implicit assumption by symmetry considerations with respect to geometry, loading and response by which only 1/4 of the shell is modelled. Lately [4,5] noticed that the symmetric solution was incorrect and concluded by numerical analyses and related experiments that there exists an asymmetric solution in terms of bifurcation at a lower load than the load limit point for the symmetric solution.

At a bifurcation point the system has multiple solutions and a secondary equilibrium path may exists which at the point of bifurcation branches away from the fundamental path (the equilibrium path for the perfect system). Bifurcation points are commonly predicted by either linear prebuckling analysis or geometrically nonlinear analysis of a slightly distorted imperfect system which may be accomplished by introducing a geometric imperfection in the form of the first linear buckling mode with some prescribed amplitude. Such a bifurcation point exists for this model and occurs approximately 12% below the load limit point for the symmetric solution, thus this is the preferred lower energy path.

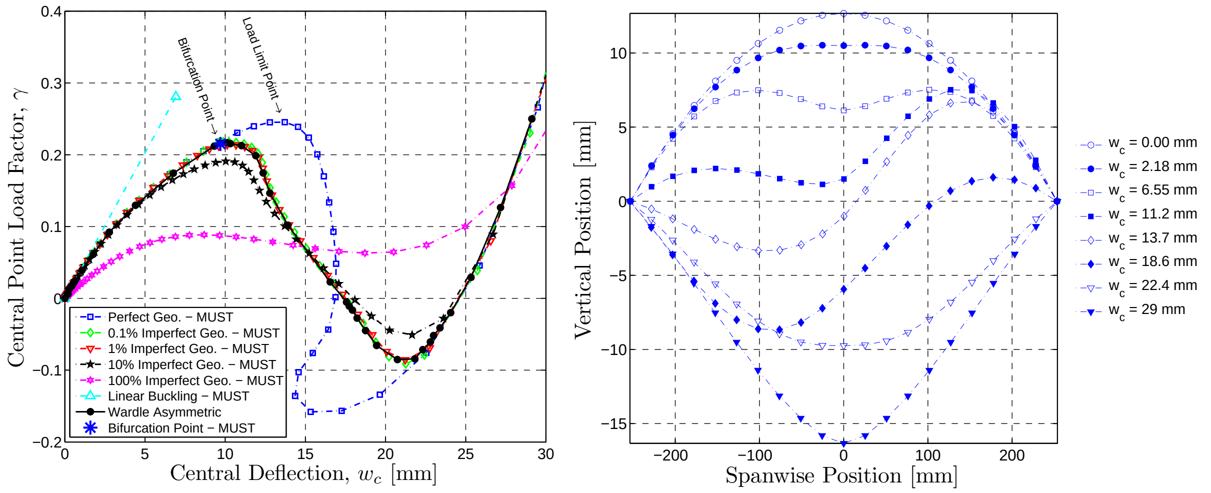

Figure 4. Left: Load-deflection response solutions of the perfect and imperfect systems together with the linear prebuckling solution. Right: Central spanwise mode shapes at different values of center deflection, w_c, obtained by FEA on the imperfect system with imperfection amplitude of 0.1%.

Linear prebuckling analysis yields a very poor prediction of the bifurcation point which is caused by the inherent assumption that the structure is assumed to behave linearly up until the buckling point. Several imperfect systems are analyzed with different amplitudes where the imperfection amplitude is defined as the largest translational component of the first linear buckling mode with respect to the shell thickness. The use of imperfections as a method to discover bifurcation points and associated branches may not always be trustworthy. It can be observed by the equilibrium paths of the imperfect systems in Fig. 4 that the imperfection amplitude has to be lower than approximately 1% in order not to change the problem and thereby the solution of the original problem which demonstrates the difficulty in discovering bifurcation points, i.e. the imperfection amplitude has to be large enough to induce bifurcation but also small enough so as not to change the problem. The same difficulties apply in the selection of the imperfection mode. The two reliable imperfect equilibrium paths (0.1% & 1% Imperfect Geo.) show a limit point in the region of the bifurcation point and do not exhibit a deflection limit point but rejoin the equilibrium path with the symmetric response at large values of center deflection. In this region the response is dominated by membrane stretching with symmetric modes, see Fig. 4 right. [4,5] obtained an almost identical solution by introducing imperfections with the so-called asymmetric meshing technique (AMT) and subsequent geometrically nonlinear analysis, see Fig. 4.

In order to determine the bifurcation load precisely and classify the type of bifurcation, a geometrically nonlinear analysis upon the perfect structure is conducted with a very small step size. With such a small step size it is possible to trace the branching from the fundamental to the secondary bifurcated path, as shown in Fig. 5, without changing the system with imperfections. The solution is completed with MUST and verified by a similar model in ANSYS where 9-noded shell elements have been applied to model the cylindrical shell example.

The bifurcation point is accurately determined at a load level of *526N* and it is quite clear that the bifurcation point is unstable, i.e. in load control the structure will at the bifurcation point experience a dynamic snap-through onto a stable configuration which is located on the fundamental equilibrium path.

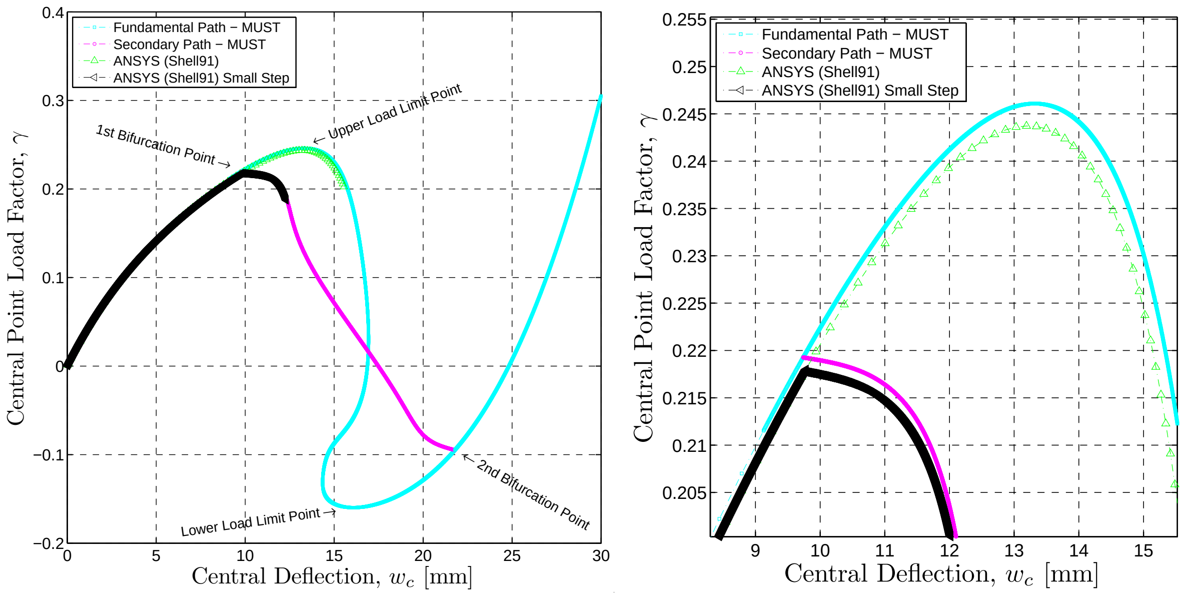

Figure 5. Left: Load-deflection curves of the perfect symmetric system for both the fundamental equilibrium path and the secondary bifurcated path obtained by MUST and a shell finite element model in ANSYS. Right: Zoomed view of the bifurcation and limit point. It is clear that the bifurcation point in unstable, i.e. the tangent is negative directly after bifurcation.

These results disprove the results published in [4,5] and also reported in [8] in which it is concluded that the bifurcation point is stable. [4,5] conclude that the bifurcation point is stable, i.e. the bifurcated path is stable and that the structure is able to carry more load until a load limit point on the bifurcated path is reached. This is not correct but probably just a wrong interpretation of the numerical results. The asymmetric solution in [4,5] is obtained by geometrically nonlinear analysis of an imperfect system. It is correct that the stability limit of the equilibrium path for the imperfect system is characterized by a limit point but the bifurcation point for the imperfect system is non-existing. The unstable bifurcation point of the perfect system is merely transformed into a limit point for the imperfect structure. Thus, this benchmark example clearly demonstrates that buckling analysis of immediate simple structures still represents a challenging task.

5 Nonlinear buckling optimization of composite cylindrical shell

The starting point for a reliable nonlinear buckling optimization procedure is the ability to evaluate the point of stability with reasonable precision. As exemplified by the proceeding numerical example, linear prebuckling analysis is not valid for determining buckling of general type and in cases where geometric nonlinearity cannot be ignored, thus geometrically nonlinear analysis is required. Furthermore, the analysis procedure should be able to handle and discover bifurcation as well as limit point instability, depending on what type of stability is first to arrive on the equilibrium path.

It is desirable only to analyse the perfect structure and not apply imperfections as a method for predicting bifurcation points due to the problems in selecting a reasonable imperfection mode and amplitude. During optimization the stability type may even change and the chosen imperfection for the system may no longer be valid for inducing the structure to bifurcate. Finally, in case of a stable bifurcation point the buckling point may simply disappear with the introduction of imperfections and thereby not be identified during the analysis.

During optimization, mode or stability type switching may occur, i.e. the first buckling point to arrive on the equilibrium path may change from a bifurcation point to a limit point or vice versa, and should be considered in the optimization formulation.

For effective treatment of the optimization problem it should be formulated as a mathematical programming problem that is solved by gradient-based optimizers, thus design sensitivities must be derived and calculated in an efficient way. This will be outlined in the presentation at ECCM15 and optimization results with the developed optimization method will be provided.

6 Conclusions

There is a need for development of an integrated approach that reliably optimizes structures with respect to a general type instability, i.e. simultaneously handles bifurcation and limit point instability, and especially in cases where geometrically nonlinear effects cannot be ignored. This study addresses these issues and presents a unified optimization procedure that solves these problems. This allows the material utilization of buckling critical laminated structures to be pushed to the limit in an efficient way in order to obtain lighter and stronger structures. The findings of this work have lately been published in [9].

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