NEW COMPOSITE BEAM THEORY INCLUDING TORSIONAL-WARPING SHEAR DEFORMATIONS

L.P. Kollár* and A. Pluzsik

Department of Mechanics, Materials and Structures,
Budapest University of Technology and Economics, Műegyetem rpt 3., Budapest, Hungary
**lkollar@eik.bme.hu*

Keywords: composite, beam theory, torsion, stiffness matrix.

Abstract

Composite beams and columns are analyzed and designed either with explicit beam expressions or with numerical (e.g. FE) methods, both require the knowledge of the cross sectional properties, i.e. the bending-, the shear-, the torsional-, warping-, axial stiffnesses and the coupling terms. These properties are calculated either by using kinematical relationships (e.g. cross sections remain-plane after the deformation of the beam) or by asymptotic methods, however in both cases the accuracy depends on the assumed degree of freedom of the model. These assumptions may lead to inaccurate or contradictory results. In this paper a new theory is presented in which no kinematical assumption is applied, rather the properties are derived from the accurate (three dimensional) equations of beams using limit transition. The theory includes both the in-plane and the torsional-warping shear deformations. As a result of the analysis the stiffness matrix of the beam is obtained which is needed for either analytical or numerical (FE) solutions.

1 Introduction

When composite beams and columns are designed the stresses and strains are calculated, and the buckling loads and in the natural frequencies are determined. In the analysis the cross sectional properties (the beam's stiffness matrix) must be known. Its calculation can be more complex for composite beams than for isotropic ones, because of the substantial differences in their response to external loads. The behaviour of isotropic and composite beams is illustrated below with the example of a thin walled I-beam.

When an *isotropic* beam is subjected to tension or bending the cross sections remain plane (Figures 1a and b), while under torsion cross sections warp (Figure *1*c).

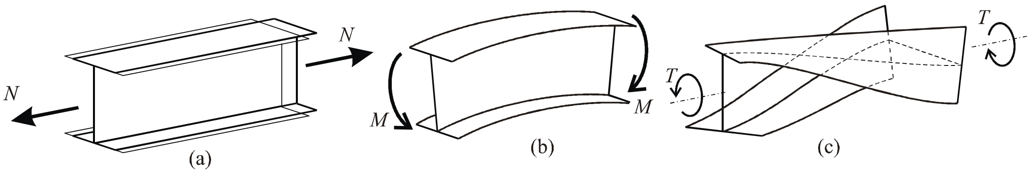

Figure 1. Deformation of an isotropic I-beam subjected to (a) tension, (b) bending and (c) torque

In the presence of structural constrains the problem of torsion is more complex and – for a built in I-beam – can be illustrated as the loading of the flanges by a force couple (Figure 2a).

Due to these loads the flanges undergo bending deformations (when the shear deformations of the flanges are neglected, Figure 2a) and shear deformations (Figure 2b). Note that for long beams the shear deformations of the flanges are negligible. For isotropic beams the tension, bending and torsion are uncoupled.

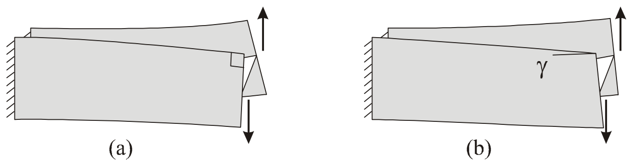

Figure 2. Deformations of a built-in isotropic I-beam subjected to torsion when (a) the shear deformations of the flanges are neglected and (b) due to the shear deformations of the flanges

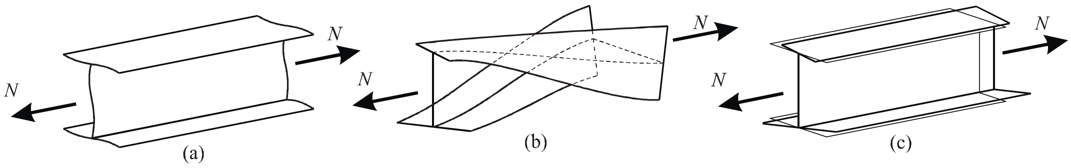

Figure 3. Possible deformations of a composite I-beam subjected to tension. (a) the warping of the cross section, (b) the tension-torsion coupling and (c) the shear deformation of the flanges

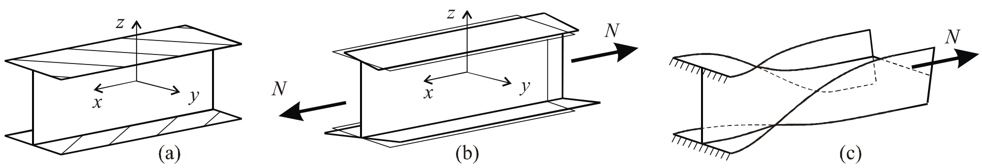

Figure 4. a) Example of an I-beam with unbalanced flanges. (b) Under pure tension there is no twist, however, the cross sections warp. (c) When one end is fixed, the beam subjected to tension will twist.

For thin-walled *composite beams* the following phenomena may occur which are significantly different from the behaviour of isotropic beams: the cross section may warp under pure tension or pure bending (the first one is illustrated in Figure 3a), there are tension-torsion, tension-bending and bending-torsion coupling (the first one is illustrated in Figure 3b), there is a tension-shear coupling in the flanges as illustrated in Figure 3c.

We further illustrate the difference between isotropic and composite beams when both tension-shear coupling (Figure 3c) and structural constrains are present. We consider the example of an I-beam with unbalanced flanges (e.g. the fibers in the upper flange are in the +45°, while the fibers in the lower flange are in the -45° direction, Figure 4a). Under pure tension the flanges undergo shear deformation (the cross section warps), however, there is no twist along the beam (Figure 4b) and there is no tension-twist coupling. When one end is built-in and the other end is free, under tension the rotation of the beam will be significant due to constrained warping (Figure 4c). (When $\overline{GI_t}$ is negligible the entire beam will undergo a uniform rate of twist.) Note that this effect is significant also for long beams.

Isotropic beam theories and the corresponding computer codes can take into account all the effects explained in Figures 1 and 2 except the shear deformation due to torsion and structural constraint (Figure 2b), note, however, that for isotopic beams this effect is negligible.

Anisotropic beam theories (and the corresponding computer codes) can handle the warping of the cross sections (Figure 3a), which means that cross sections do not remain plane under pure tension and bending, and also the coupling among tension, torsion and bending, however they do not include

- the shear deformation of the flanges due to restrained warping (Figure 2b),
- the tension-shear coupling in the flanges (Figure 3c).

We emphasize that for composite beams the restrained warping induced shear deformations are important and – except for long beams – can not be neglected [1]. In addition, when tension-shear coupling is present (Figure 4b and c) theories which do not include this effect may lead to unacceptable results even for long beams.

In this paper we present an analysis of thin-walled composite beams, taking into account restrained warping induced shear deformation and the tension-shear coupling. We will summarize only the basic idea of the method to calculate the *stiffness matrix* of composite beams, the details of the analysis was presented recently by the authors [2].
Beam theories give the relationships between the *displacements* of the beam axis, generalized *strains*, internal *forces* and *loads*. These relationships are given by the strain-displacement relationships (geometrical equations), material law (constitutive equations) and the equilibrium equations. These are written as:

$$\boldsymbol{\varepsilon} = \hat{\boldsymbol{\Theta}}\mathbf{u}, \qquad \mathbf{N} = \mathbf{M}\boldsymbol{\varepsilon}, \qquad \mathbf{p} = \hat{\boldsymbol{\Theta}}^{*}\mathbf{N}, \tag{EQ1}$$

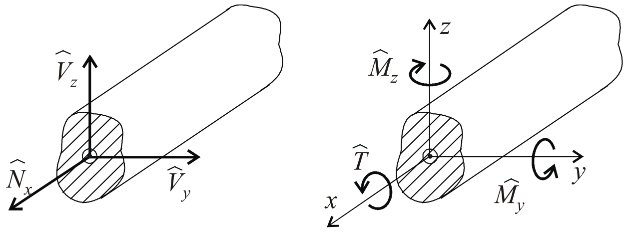

Figure 5. The stress resultants (internal forces).

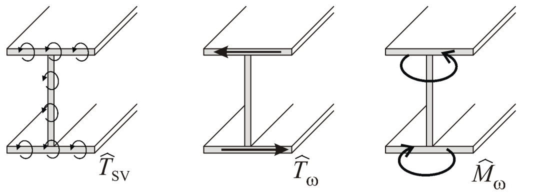

Figure 6. Illustration of Saint Venant torque, restrained warping induced torque and bimoment on an I-beam.

where $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\mathbf{N}$, $\mathbf{p}$ are the vectors of the displacements, generalized strains, internal forces and loads, respectively. $\mathbf{M}$ is the (symmetric) stiffness matrix, while $\hat{\boldsymbol{\Theta}}$ and $\hat{\boldsymbol{\Theta}}^{*}$ are operator matrices. For the *spatial* case the minimum number of internal forces is six due to the six stress resultants (Figure 5): the axial force, the two transverse shear forces, the two bending moments and the torque. (By neglecting the shear deformations the shear forces can be eliminated, but for composites – as it is discussed in [1] – this is not recommended.) It is well known that for open section beams the theory including these six forces only is inaccurate, and the torque must be divided as Saint Venant and restrained warping induced torque: $T_{SV} + T_{\omega}$ (Figure 6), where the latter one is the derivative of the bimoment (or moment couple), $T_{\omega} = \partial M_{\omega} / \partial x$. The equations of this theory – often referred to as Vlasov's theory – are given in the top part of Table 1. (The matrix $\hat{\boldsymbol{\Theta}}^{*}$ is not given, however, it can be obtained directly from the transpose of $\hat{\boldsymbol{\Theta}}$, if the signs of the first derivatives are reversed.) In this theory six displacements must be taken into account (see Table 1, top): where u, v and w are the displacements of the axis, ψ is the rotation of the cross section about the beam's axis and χ_y, χ_z are the rotation of the cross sections about the z and y axes. For cross sections made of isotropic or orthotropic materials only some of the elements in the stiffness matrix are zero [1], which are denoted by stars in Table 1.

Classical theory

$$\left\{\begin{array}{c} \Gamma \\ \hline \varepsilon_x \\ 1/\rho_z \\ 1/\rho_y \\ \vartheta \\ \hline \gamma_y \\ \gamma_z \end{array}\right\} = \underbrace{\left[\begin{array}{cccc|cc} & & & -(\,)'' & & \\ \hline (\,)' & & & & & \\ & & & & -(\,)' & \\ & & & & & -(\,)' \\ & & & (\,)' & & \\ \hline & (\,)' & & & -1 & \\ & & (\,)' & & & -1 \end{array}\right]}_{\hat{\Theta}} \left\{\begin{array}{c} u \\ v \\ w \\ \psi \\ \hline \chi_y \\ \chi_z \end{array}\right\}, \qquad \left\{\begin{array}{c} p_x \\ p_y \\ p_z \\ t \\ \hline -m_z \\ -m_y \end{array}\right\} = \hat{\Theta}^* \left\{\begin{array}{c} \hat{M}_\omega \\ \hline \hat{N}_x \\ \hat{M}_z \\ \hat{M}_y \\ \hat{T}_{\mathrm{SV}} \\ \hline \hat{V}_y \\ \hat{V}_z \end{array}\right\}$$

$$\left\{\begin{array}{c} \hat{M}_\omega \\ \hline \hat{N}_x \\ \hat{M}_z \\ \hat{M}_y \\ \hat{T}_{\mathrm{SV}} \\ \hline \hat{V}_y \\ \hat{V}_z \end{array}\right\} = \left[\begin{array}{c|cccc|cc} \overline{EI}_\omega & \cdots & & & & & \\ \hline \vdots & \overline{EA} & & & & & \\ & & \overline{EI}_z & & & & \\ & & & \overline{EI}_y & & & \\ & & & & \overline{GI}_t & & \\ \hline & & & & & S_y & \\ & & & & & & S_z \end{array}\right] \left\{\begin{array}{c} \Gamma \\ \hline \varepsilon_x \\ 1/\rho_z \\ 1/\rho_y \\ \vartheta \\ \hline \gamma_y \\ \gamma_z \end{array}\right\}, \quad \mathbf{M}_{\mathrm{or}} = \left[\begin{array}{c|cccc|cc} * & * & * & * & & & \\ \hline * & * & * & * & & & \\ * & * & * & * & & & \\ * & * & * & * & & & \\ & & & & * & * & * \\ \hline & & & & * & * & * \\ & & & & * & * & * \end{array}\right]$$

Torsional-warping shear deformation theory

$$\left\{\begin{array}{c} \Gamma \\ \hline \varepsilon_x \\ 1/\rho_z \\ 1/\rho_y \\ \vartheta \\ \hline \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{array}\right\} = \underbrace{\left[\begin{array}{cccc|ccc} & & & & & & -(\,)' \\ \hline (\,)' & & & & & & \\ & & & & -(\,)' & & \\ & & & & & -(\,)' & \\ & & & (\,)' & & & \\ \hline & (\,)' & & & -1 & & \\ & & (\,)' & & & -1 & \\ & & & (\,)' & & & -1 \end{array}\right]}_{\hat{\Theta}} \left\{\begin{array}{c} u \\ v \\ w \\ \psi \\ \hline \chi_y \\ \chi_z \\ \vartheta^{\mathrm{B}} \end{array}\right\} \qquad \left\{\begin{array}{c} p_x \\ \hline p_y \\ p_z \\ t \\ \hline -m_z \\ -m_y \\ -m_\omega \end{array}\right\} = \hat{\Theta}^* \left\{\begin{array}{c} \hat{M}_\omega \\ \hline \hat{N}_x \\ \hat{M}_z \\ \hat{M}_y \\ \hat{T}_{\mathrm{SV}} \\ \hline \hat{V}_y \\ \hat{V}_z \\ \hat{T}_\omega \end{array}\right\}$$

$$\left\{\begin{array}{c} \hat{M}_\omega \\ \hline \hat{N}_x \\ \hat{M}_z \\ \hat{M}_y \\ \hat{T}_{\mathrm{SV}} \\ \hline \hat{V}_y \\ \hat{V}_z \\ \hat{T}_\omega \end{array}\right\} = \left[\begin{array}{c|cccc|ccc} \overline{EI}_\omega & \cdots & & & & & & \\ \hline \vdots & \overline{EA} & & & & & & \\ & & \overline{EI}_z & & & & & \\ & & & \overline{EI}_y & & & & \\ & & & & \overline{GI}_t & & & \\ \hline & & & & & S_y & & \\ & & & & & & S_z & \\ & & & & & & & S_\omega \end{array}\right] \left\{\begin{array}{c} \Gamma \\ \hline \varepsilon_x \\ 1/\rho_z \\ 1/\rho_y \\ \vartheta \\ \hline \gamma_y \\ \gamma_z \\ \vartheta^{\mathrm{S}} \end{array}\right\} \quad \mathbf{M}_{\mathrm{or}} = \left[\begin{array}{c|cccc|ccc} * & * & * & * & & & & \\ \hline * & * & * & * & & & & \\ * & * & * & * & & & & \\ * & * & * & * & & & & \\ & & & & * & * & * & * \\ \hline & & & & * & * & * & * \\ & & & & * & * & * & * \\ & & & & * & * & * & * \end{array}\right]$$

Table 1. Beam equations for the spatial problem (Prime denotes derivative with respect to *x*. Stars in the $\mathbf{M}_{\mathrm{or}}$ matrix show the nonzero elements in the stiffness matrix of an orthotropic beam.)

Shortcomings of the classical theory. It is well known that composite structures undergo higher shear deformation than structures made of conventional materials, and hence the shear deformation should not be neglected. For restrained warping the torsional shear deformations must also be taken into account. This is illustrated in Figure 2 for an orthotropic I beam. According to Vlasov's theory, when an I beam is subjected to torsion there is no shear deformation of the flanges (Figure 2a), when, in fact, the shear deformation (illustrated in Figure 2b) may be significant. This effect can be modelled by the *torsional-warping shear deformation theory*. It can be argued that for longer beams this effect is negligible. This may be true for orthotropic beams, but not for anisotropic ones, as it was shown in Figure 4.

2 Torsional-warping shear deformation theory

To understand the torsional-warping shear deformations, first we consider the case when a *beam deforms only in a plane* (e.g. in the *x-y* plane). In the classical beam theory [3], (when the shear deformation is neglected), the displacements of the axis in the *x*- and *y* directions (*u* and *v*) are used to calculate the strains and deformations of any point of the cross section. When the shear deformation is taken into account, according to Timoshenko's beam theory (see Kollár and Springer [1] for composite beams), three displacement functions of the axis are required: the displacement along and perpendicular to the axis (*u* and *v*) and the rotation of the cross section (χ_y). In other words, the slope of the displacement consists of two parts, the rotation of the cross section and the shear strain:

$$\frac{\partial v}{\partial x} = \chi_y + \gamma_y . \tag{EQ3}$$

This is illustrated in Figure 7b.

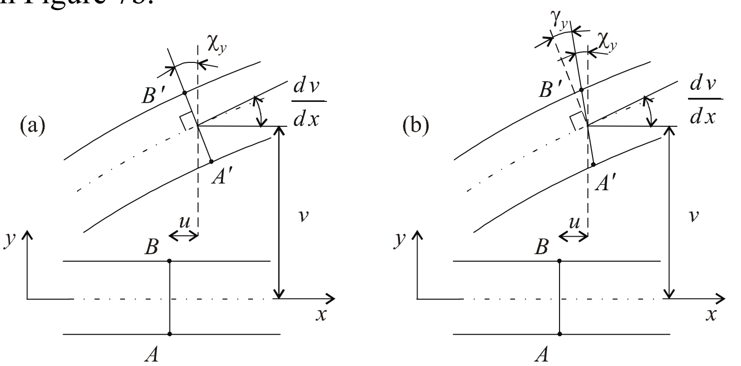

Figure 7. In-plane deformations of a beam (a) without and (b) with shear deformations.

When a *beam is subjected to torsion*, in the classical (Vlasov or Wagner) theory only the rotation of the cross section (ψ) about the beam's axis is used [3], [4] to calculate the displacements of any point of the cross section. When the axial warping is constrained, an open section beam carries the torque load mainly by the bending and shear of the flanges, as illustrated in Figure 2 for a symmetrical I-beam. Note, however, that according to Vlasov's theory the shear deformations of the walls (Figure 2b) are neglected. To overcome this shortcoming, analogously to Timoshenko's beam theory, we introduced [5] a new displacement function (in addition to the rotation of the cross section, ψ): the rate of twist due to warping (ϑ^{B}). In other words, the rate of twist ($\partial\psi / \partial x$) consists of two parts, one when the shear deformation is zero and one when the warping is zero:

$$\frac{\partial \psi}{\partial x} = \vartheta^{\mathrm{B}} + \vartheta^{\mathrm{S}} . \tag{EQ4}$$

We must give credit to Wu and Sun [6], who suggested first the introduction of this new function.

In summary, in this theory seven displacements must be taken into account:

$$u,\quad v,\quad w,\quad \psi,\ \chi_y,\ \chi_z,\ \vartheta^{\mathrm{B}} \tag{EQ5}$$

where u, v and w are the displacements of the axis, ψ is the rotation of the cross section about the beam's axis, χ_y, χ_z are the rotation of the cross sections about the *z* and *y* axes, and ϑ^{B} is the rate of twist due to warping. This theory was developed for *orthotropic* open [5] and closed [7] section beams, the stiffeness matrix is given in Figure 8.

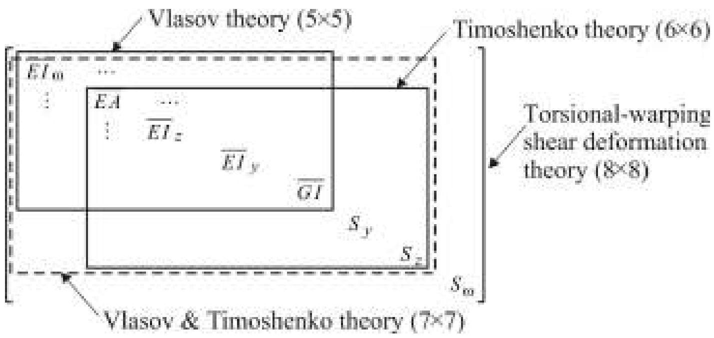

Figure 8. Stiffness matrix of Vlasov-, Timoshenko- and torsional-warping shear deformation theory.

The new element in the stiffness matrix is the rotational shear stiffness S_ω. For a few cases analytical expressions are given for the calculation of the stiffnesses of orthotropic composite thin walled beams including the rotational shear stiffness [5],[7].

The equations of the torsional warping shear deformation theory are summarized in Table 1.

3 Methods to calculation of the stiffnesses

For anisotropic beams there are 8 × 8 elements of the (symmetric) stiffness matrix if the torsional-warping shear deformation theory is used. There are several methods, which can be used to determine the stiffnesses of a beam, the most important ones are discussed below.

The most common approach of beam theories that the cross sectional distributions of the displacements (or stresses, strains) are approximated by given functions. This is the basic idea of the classical Bernoulli theory, the Timoshenko beam theory, or Vlasov's theory. The literature of this approach was summarized in [2]. The most accurate and powerful method of Jung et al. [8] is based on the combination of the first order shear theory and torsional-warping deformations (Vlasov and Timoshenko theory), so it results in a 7×7 full stiffness matrix (Figure 8) for open section anisotropic beams.

Unfortunately, only for simple cross sections and layups can the correct distribution of stress or strain field be assumed a priory. For arbitrary layup, the distribution of the cross sectional displacement is rather complex, and hence a different approach is needed. In variational *asymptotic approach* the displacement approximations are refined in an iterative manner. The mathematical basis can be found in Antman [9]. Finite element codes, NABSA and VABS relying on this method are worked out by [10], [11], [12]. The most adequate asymptotic beam theory, VABS (Variational Asymptotic Beam Section Analysis) is worked out by Hodges [13].

In VABS refined beam models can be used: For open section beams either the Timoshenko theory or the Vlasov theory is applied [13], [14] which results either in a 6×6 or a 5×5 stiffness matrix (Figure 8). For closed section beams the Timoshenko theory is applied [13] and [14] which results in a 6×6 stiffness matrix (Figure 8). The solution of VABS is valid for arbitrary cross sections and also for initially curved and twisted beams [13], however, it contains neither the restrained warping induced shear deformations nor the effect of restrained warping for closed cross sections.

4 Method of Solution

The basic idea of calculating the stiffenesses is as follows. The displacements are assumed in the form of sine (or cosine) functions: $\sin\alpha x$, $\cos\alpha x$ ($\alpha = \pi / L$). Then the strains, internal forces and loads are determined. Each contains either sine or cosine functions only. Now the average strain energy per unit length of the structure is determined for the beam model and also for the 3D model. Both are function of $1/L$. The stiffness matrix $\mathbf{M}$ is determined from the condition that the coefficients of the Taylor series expansions of the strain energies are equal. The key of the solution is that for trigonometrical functions the differential equation system can be replaced by ordinary (matrix) equations: If $\sin\alpha x$, $\cos\alpha x$ $(\alpha = \pi / L)$ is differentiated with respect to x the result is a trigonometrical function multiplied by α. As a consequence, for trigonometrical displacements the differential equation system Equation (EQ1) can be replaced by ordinary equations, where the coefficient matrices contain α. The details of the solution is given in [2].

5 Numerical examples

We developed a computer code, designated as BEAMSIN to calculate the stiffnesses of thin walled anisotropic beams. The code is based on the 3D analytical solution of beams presented in [2]. When the stiffnesses are known the displacement can be calculated either numerically (FE) or analytically [2]. Here only the results of two examples [2] are presented.

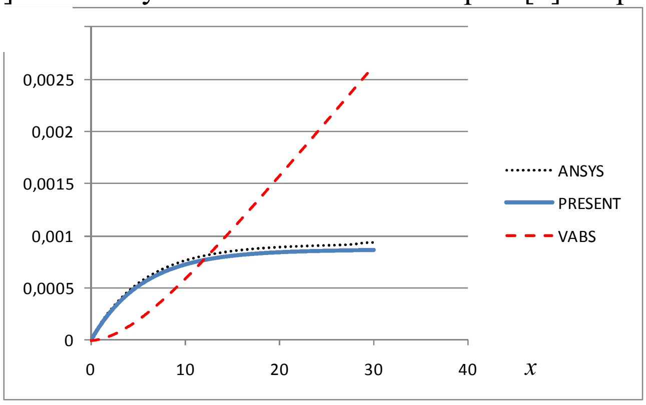

Figure 9. Rotation ψ of an I-beam cantilever with unbalanced layups subjected to tensile force

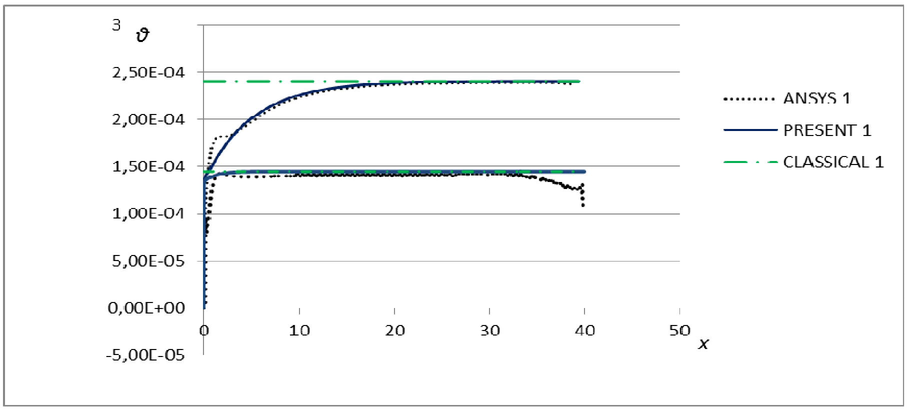

Figure 10. Rotation ψ of a box-beam cantilever subjected to a torque load

The first one is an I-beam cantilever with unbalanced flanges subjected to a tensile load (Figure 4c), which causes the rotation of the cross section. These calculations were compared to the results of a finite element (ANSYS) analysis, where shell elements were used. The solutions of the VABS analysis [13], [14] are also included. VABS does not contain the restrained warping induced shear deformations, hence it cannot predict well the tension – warping-shear coupling. The second example is a box beam cantilever subjected to a torque at the end. A "Classical" beam theory which does not contain the restrained warping effects give constant rate of twist along the beam length ($\vartheta = T / GI_t$). The results of the present and the "classical" theory and the finite element calculation are plotted in Figure 10 for two cross sections. When the stiffnesses of the walls are identical (bottom curves) the effect of restrained warping is negligible. When the stiffnesses of the wall differ from each other significantly (top curves) the restrained warping play a significant role in the deformations.

6. Discussion

We presented a new method to determine the stiffnesses of anisotropic beams without assuming kinematical relationships. The applied theory contains the restrained torsional warping (Vlasov theory), the in plane shear deformations (Timoshenko theory) and the torsional-warping shear deformations. According to our knowledge none of the previous theories contained all these effects for anisotropic beams. These effects may be neglected for long beams with balanced layup, however omitting the new term (i.e. the torsional warping shear stiffness) may lead to unacceptable results for short beams and also for long beams, when the layup is unbalanced.

Acknowledgement. This work was supported by the Hungarian Scientific Research Fund (grant number K-77803).

References

[1] Kollár LP, Springer GS. Mechanics of Composite Structures. Cambridge: Cambridge University Press, 2003, p.480.

[2] Kollár LP, Pluzsik A. Bending and Torsion of Composite Beams (Torsional-warping Shear Deformation Theory). *Journal of Reinforced Plastics and Composites* 2012; 31:441-480.

[3] Megson THG. Aircraft Structures for Engineering Students. New York: Halsted Press, John Wiley & Sons. U.S. 1990.

[4] Vlasov VZ. Thin Walled Elastic Beams. Washington: Office of Technical Services, U.S. Department of Commerce, 25, DC, TT-61-11400 1961.

[5] Kollár LP. Flexural-torsional Buckling of Open Section Composite Columns with Shear Deformation. *International Journal of Solids and Structures* 2001; 38:7525-7541.

[6] Wu X and Sun TC. Simplified Theory for Composite Thin-Walled Beams. *AIAA Journal* 1992; 30:2945-2951.

[7] Pluzsik A and Kollár LP. Torsion of Closed Section Orthotropic Thin-Walled Beams. *International Journal of Solids and Structures* 2005; 43:5307-5336.

[8] Jung SN, Nagaraj VT and Chopra I. Refined Structural Model for Thin- and Thick-Walled Composite Rotor Blades. *AIAA Journal* 2002; 40: 1:105-116.

[9] Antman SS. The Theory of Rods. In: Antman SS. Mechanics of Solids Vol. II. Berlin: Springer-Verlag, 1984, pp. 641-703.

[10] Giavotto V, Borri, M, Mantegazza P, Ghiringhelli G, Carmaschi V, Maffioli GC and Mussi F. Anisotropic beam theory and applications. *Computers and Structures* 1983; 16: 403-413.

[11] Yu W, Hodges DH, Volovoi V and Fuchs ED. A Generalized Vlasov Theory for Composite Beams. *Thin-Walled Structures* 2005; 39: 5101-5121.

[12] Yu W, Volovoi V, Hodges DH and Hong X. Validation of the Variational Asymptotic Beam Sectional Analysis (VABS). *AIAA Journal* 2002; 40: 10:2105-2113.

[13] Hodges DH. Nonlinear Composite Beam Theory. U.S.: American Institute of Aeronautics and Astronautics 2006, p.300.

[14] Yu W, Hodges DH, Volovoi V and Cesnik, CES. On Timoshenko-like modeling of initially curved and twisted composite beams. *International Journal of Solids and Structures* 2002; 39:5101-5121.