ANALYSIS OF ELASTIC FIELDS IN AN ISOTROPIC MEDIUM CONTAINING A PENNY-SHAPED CRACK BY THE RITZ METHOD

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Keywords: Penny-shaped crack; Stress intensity factor; Fracture mechanics; Ritz method

Abstract

In the crack growth analysis, the Stress Intensity Factor (SIF) is a fundamental prerequisite. In the present study, the mode I stress intensity factor (SIF) of three-dimensional penny-Shaped crack is obtained in an isotropic elastic cylindrical medium with arbitrary dimensions under arbitrary loading at the top of the cylinder, by the semi-analytical method based on the Rayleigh-Ritz method. This method that is based on minimizing the potential energy amount of the whole of the system, gives a very close results to the previous studies. Defining the displacements (elastic fields) by hypothetical functions in a defined coordinate system is the base of this research. So for creating the singularity conditions at the tip of the crack the appropriate terms should be found.

1 Introduction

One of the subjects recently used in the study of failure of structures and blocks under alternation loads, is the fracture mechanics and the study of now failure parameters influence structures. The crack growth and development rate depends on material toughness and amount of energy release, that is if the stress intensity factor (SIF) oversteps a certain amount on the tip of the crack, the crack will grow. In this study we examine the elastic fields which include stress and strain and accordingly the stress intensity factor (SIF) would calculated on the tip of the crack.

1.1 Previous literature

For a through finding on the SIFs of cracks in the literature, up to the year 2000, one should refer to the handbooks of Murakami [1] and Tada et al. [2]. A close scrutiny of the literature reveals that, except for a very few specific cases, the exact solution to the mode I SIF of three-dimensional penny-shape and elliptic cracks under polynomial loading at infinity has obtained by Shodja and Ojaghnezhad [3]. Most of the closed-form solutions to the SIF pertinent to a penny shape crack in an infinite isotropic elastic body are devoted to at most linear far-field loading. The more general geometry of an elliptic crack under a uniform far-field tension was considered by Irwin [4] employing the stress function theory.

1.2 Ritz method

Hereunto, the tridimensional cracks were mainly studied using the finite element-method, which results in much cost and time consumption, because of too element division and difficulty at defining tridimensional singular element. On the other hand, the need for more

efficient method is felt, regarding inability of laboratory methods in the study of tridimensional cracks. The purpose of this study is to achieve more inexpensive and rapid methods other than existing methods like finite-element method.
Ritz method follows the principle of system's total potential energy constant. The base is to define point's transformation in the form of imaginary functions in the systems defined coordinates and to minimize the systems total potential energy. One of the features of this method is the possibility of using local coordinates. In this method, at first we write the function of the system's total potential energy. An imaginary function is then determined for displacement, such that satisfy the geometric border conditions (including displacements and gradients). Imaginary function is placed to the potential equation and integrated on the structure dimensions under study. The obtained potential function is then minimized towards unknown parameters, thus we obtain the "**n** equation and **n** unknown factot" system, by solving these equations we may represent the shape function.

1.3 Penny-shaped crack
Consider an ellipsoidal inhomogeneity in a tridimensional environment (Fig.1). It is called crack in the situation that elastic moduli of inhomogeneity is zero [3].

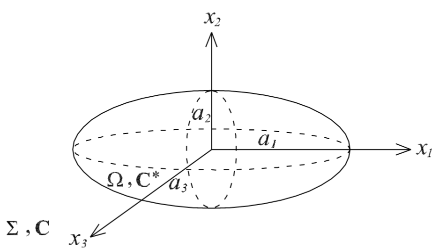

Fig 1. The ellipsoidal inhomogeneity

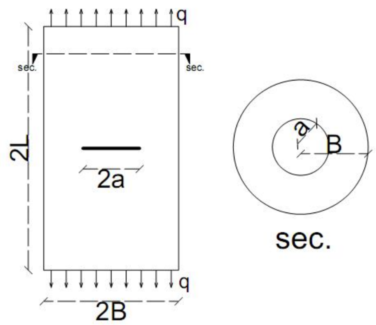

Fig 2. an isotropic elastic cylindrical medium with a penny-shaped crack

According to the Fig.1, when $a_3 \to 0$ and $a_1=a_2=a$, this crack is defined as a penny-shaped crack with radius a. The problem under study in this paper is defined as an isotropic elastic cylindrical medium with length of 2L, and diameter of 2B, wide q loading along the length of cylinder and in the cylinder's 2 ends sides (Fig.2). Penny-shaped crack with diameter of 2a is located at the center of cylinder. The most important mode of Penny-shaped crack failure, is mode I (opening mode). In this paper, penny-shaped crack is under mode I loading.

2 Solving the problem of cylindrical medium with penny-shaped crack
According to existing symmetry in the cylindrical medium of Fig.2, we consider the medium as Fig.3, to solve the problem easily. A local polar coordinates (R,φ) has been defined on the tip of the crack, to write the relations (Fig.3).

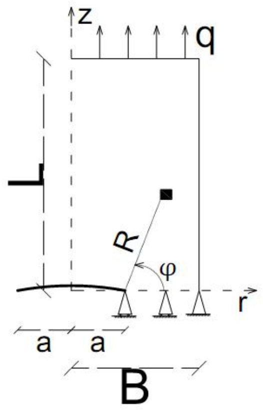

Fig 3. A medium with local polar coordinates (R,φ)

Displacement equation, in both axis **z** and **r** is considered as Eq.1. These functions (Eq.1) satisfy the geometric bordered conditions which include: ($u_z=0$ at $\varphi=0$, $0<R<(B-a)$ & $u_r\neq 0$ at $R=0$). Stress singularity on the tip of the crack, has been insured by $\sqrt{R}$. Each of the equations in the Eq.1, has the (N+1) unknown coefficient (A and B) and N is obtained from Eq.2. Generally, there are (2N+2) unknown coefficient.

$$U_z = \sum_{i+j=0}^{p} \sqrt{R} A_{ij} R^i \sin\left(\frac{2j+1}{2}\varphi\right) + A_0 R \sin(\varphi) \quad \text{(a)}$$

$$U_r = \sum_{i+j=0}^{p} \sqrt{R} B_{ij} R^i \cos\left(\frac{2j+1}{2}\varphi\right) + B_0 \left(R\cos(\varphi) + a\right) \quad \text{(b)}$$

(Eq.1)

$$N = \frac{(p+1)(p+2)}{2} \quad \text{(Eq.2)}$$

We define the equation of systems total energy as Eq.3, in which U and W are systems strain energy and the work obtained from external force respectively.

$$\Phi = U - W \quad \text{(Eq.3)}$$

2.1 Strain energy (U)

Regardless of the strain energies, resulting from axial and shearing loads, U is defined as Eq.4, in which σ an ε are stress and strain respectively, and integration is done in the whole of medium.

$$U = \frac{1}{2}\iiint \sigma.\varepsilon dV \quad \text{(Eq.4)} \qquad \sigma.\varepsilon = \sigma_r\varepsilon_r + \sigma_\theta\varepsilon_\theta + \sigma_z\varepsilon_z + \tau_{rz}\gamma_{rz} \quad \text{(Eq.5)}$$

σ.ε is written as Eq.5, according to strains relations in the cylindrical coordinates and also problem conditions ($u_\theta=0$). Stresses are defined as Eq.6, according to the elastic relations in the isotropic medium, thus it's possible to solve the Eq.4. To integrate in the whole of medium, according to singularity of stresses on the tip of the crack, we have used the Duffy transformation method, which exists in the reference number [5].

$$\begin{cases} \sigma_r = c_{11}\varepsilon_r + c_{12}\varepsilon_\theta + c_{13}\varepsilon_z \\ \sigma_\theta = c_{21}\varepsilon_r + c_{22}\varepsilon_\theta + c_{23}\varepsilon_z \\ \sigma_z = c_{31}\varepsilon_r + c_{32}\varepsilon_\theta + c_{33}\varepsilon_z \\ \tau_{rz} = c_{66}\gamma_{rz} \end{cases} \quad \text{(Eq.6)}$$

2.2 External force work (W)

In this problem, W is obtained from Eq.7, according to the loading along axis z. integration is done on the loading surface (z=L).

$$W = \int U_z . q(r) dA \quad \text{(Eq.7)}$$

2.3 Solving the equations

To solve the existing integrals in the W and U equations, we have used the numerical method of Gauss-legendre, which exists in reference number [6]. To do this, we have done the programing in FORTRAN90. In this step of Ritz method, system's total potential energy is minimized toward unknown coefficients in Eq.8, in which α:(A,B). If Eq.8 be written in the form of matrix, Eq.9 will obtain, in which [K] is the matrix obtained from dU/dα, and [X] is the matrix of unknown coefficients and [F] is the matrix obtained from dW/dα. [K] is a squared matrix, with dimensions (2N+2), [X] and [F] are matrix with dimension (2N+2)*1.

$$\frac{\partial \Phi}{\partial \alpha} = \frac{\partial U}{\partial \alpha} - \frac{\partial W}{\partial \alpha} = 0 \Rightarrow \frac{\partial U}{\partial \alpha} = \frac{\partial W}{\partial \alpha} \quad \text{(Eq.8)} \qquad [K] \times [X] = [F] \quad \text{(Eq.9)}$$

By differentiation from V and W (Eq.4, Eq.7), and replacing the relating terms in the matrix, unknown coefficient matrix, [X], is obtained by solving the Eq.9. By replacing the obtained unknown coefficient of matrix [X] in the Eq.1, the displacement field and also stress field will obtain.

2.4 stress intensity factor (SIF)

Irwin has defined the stress intensity factor (SIF) on the tip of the crack as Eq.10. In this paper we calculate the amount of stress in the points with coordinates (R=1e-10, φ=0), along the axis z, as approaches the tip of the crack, and by replacing it in the Eq.10, the mode I SIF, will obtain from Eq.11.

$$K_1 = \lim_{R \to 0} \sqrt{2\pi R} \times \sigma_z \quad \text{(Eq.10)} \qquad K_1 = \sqrt{2\pi R} \times \sigma_z \quad \text{(Eq.11)}$$

3 Results

We have solved the problem in an isotropic medium with elastic moduli E=1 kg/cm, and poisson's ratio ν=0.3. The problem dimension has been selected, according to Fig.2, by the amounts of a=0.5 cm, B=5.0 cm, L=10 cm. The problem is under loading q=1.0 kg/cm^2. We have solved the problem by changing p in the eq.1, from p=2 to p=10.

*3.1 Result's diagram*s

We have done the programing in the *Compaq visual fortran (2006).* The program includes 40 functions and 2 subroutines and in general, we have done 700 lines coding. The relating results have been shown in Fig.4 and Fig.5. As it shown in Fig.4, by increasing the order of functions (p), the crack opening shape gradient at the center of the crack, will approaches zero, and also the answers have converged, which shows the increasing of the functions' order (p), increases the accuracy.

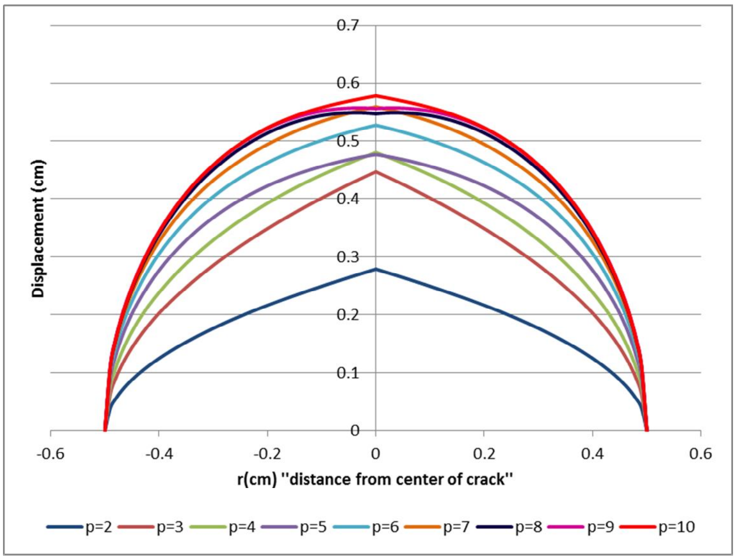

Fig 4. Crack opening shape

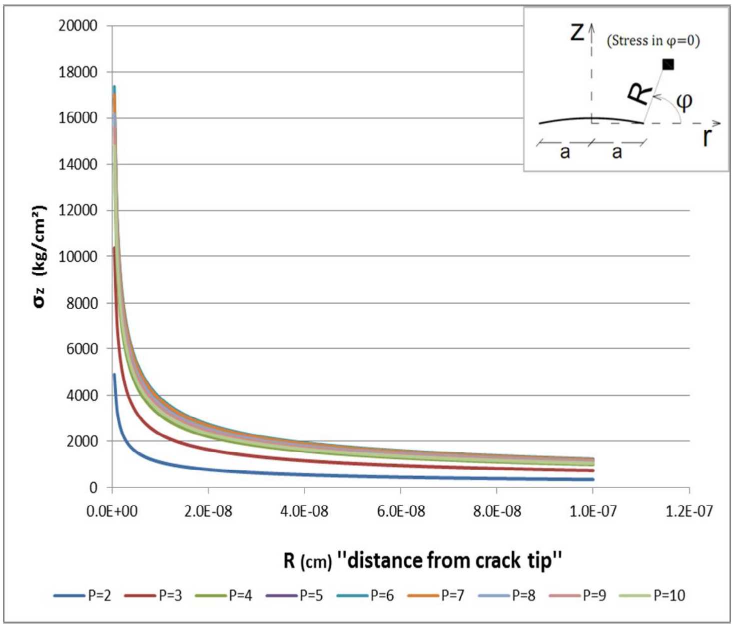

Fig 5. Stress diagram (in φ=0)

As you observed in Fig.5, by approaching the tip of the crack (R=0), the stress diagram will approach to extreme, which shows the singularity of stress on the tip of the crack. Also the diagram has converged, as the order of functions (p) increased. We have calculated the SIF, at different orders of functions, using the Eq.11 and it has shown in Fig.6. After being convergent, SIF has achieved the amount of SIF=0.83 at p=10.

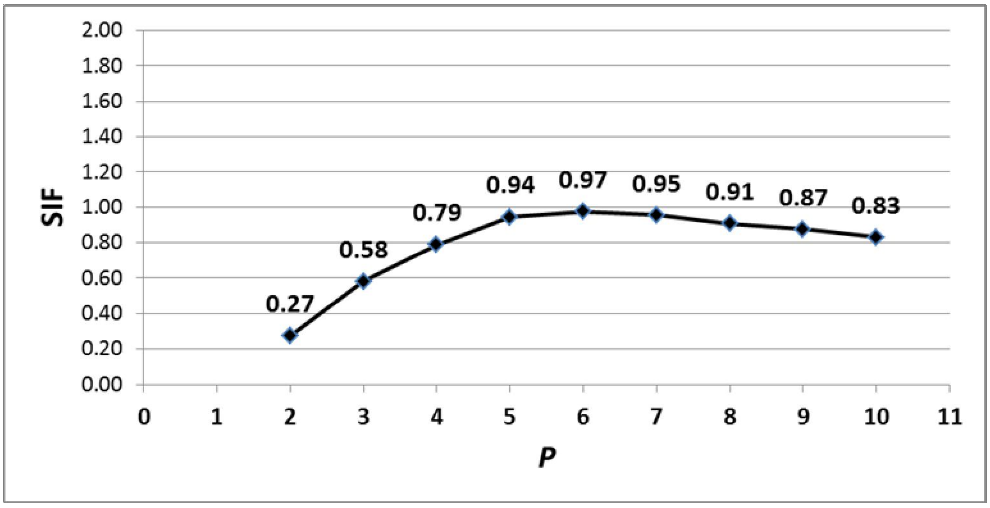

Fig 6. Stress Intensity Factor (SIF)

3.2 Results' comparing

In the Tada's handbook, this problem has been formulated for the special state of extreme medium and loading at infinity. These formulations have been shown in the Eq.12 and Eq.13. The crack opening shape, for p=10 has been compered with Tada's method, which has been shown in Fig.7. As you see, the results are very close.

$$K_1 = \frac{2}{\pi} q \sqrt{\pi a} \quad \text{(Eq.12)} \qquad\qquad V(r, z=0) = \frac{4(1-\nu^2)}{\pi E} q \sqrt{a^2 - r^2} \quad \text{(Eq.13)}$$

We have calculated the error on the center of the crack (r=0), which represent the 0.21% error at that point. We have also calculated the error in the SIF at p=10, which is 3.95 %.

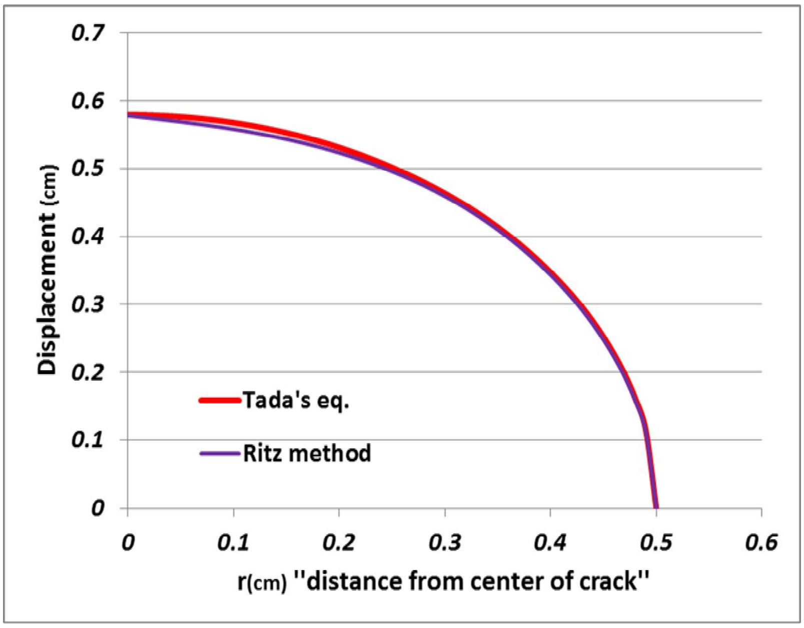

Fig 7. Crack opening shape comparision

3.3 General diagram to calculate SIF

In this section, we have presented the general diagram for different ratio of crack's dimensions to medium dimensions (Fig.7). This diagram can be used for obtaining the SIF at different dimensions of the crack. We have normalized the vertical axis in Fig.7 as (SIF/√πa).

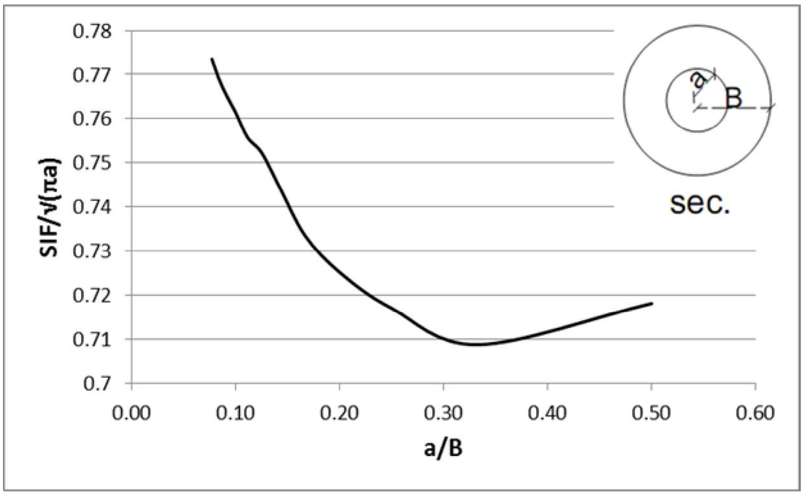

Fig 8. Crack opening shape comparison

4 Conclusion

The Ritz method is very accurate in obtaining the displacement field, but it is somewhat less accurate in stress field, because of differentiation on functions and also numerical calculations of integrals. However, since the finite-element method is cost and time consuming, this method is very applicable.

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