ON OPTIMIZATION OF COMPOSITE PLATES AGAINST BUCKLING, WITH SPECIAL ATTENTION TO BENDING-TWISTING COUPLING

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Keywords: composite plate, buckling, optimization, orientation.

Abstract
The optimization problems of maximizing the (lowest) buckling eigen value of composite plate due to layer orientations are considered. The corresponding optimality conditions for orthotropic/anisotropic plates with symmetric lay-up are studied. It is demonstrated that the bending-twisting coupling terms and the twisting moment in the principal curvature axes play important role in the optimality. In particular, for both orthotropic and anisotropic plates in case of local orientation optimization the twisting moment is equal to zero in non-umbilical points. The condition is valid "in average" in case of layer orientation angles independent of location at the plate. Numerical example of long composite plate under shear is presented and analyzed. It is demonstrated that proper choice of layer orientations leads to considerable raise of buckling level regardless of the loading direction.

1 Introduction

The investigation of anisotropic structures began in 30-ties of the last century, by numerous works devoted to plywood structures used in airplanes (see, for example, [1]). Investigating the structural stability important for the applications, it was identified that the bending-twisting terms in the buckling equation play an important role, especially in case of shear buckling.

With appearance of composites or fiber-reinforced polymers like CFRP (carbon fiber-reinforced polymer) or GFRP (glass fiber-reinforced polymer) and their wide application into aerospace, marine and civil structures the behavioral peculiarities of composite structures attracted a lot of attention of researches and engineers. During the last more than two decades a considerable number of papers dealt with buckling of composite plates subjected to compression and /or shear. It was identified that the problems were highly dependent on layer lay-up [2].

From the very appearance of composite structures the researchers investigated the corresponding structural optimization problems. In many papers the attention was paid to lay-up optimization of composite plates under buckling conditions. The majority of papers are devoted to numerical FEM-based approaches, with some attempts to use semi-analytical or analytical formulas. In such papers the search of optimum is made numerically, using genetic algorithms or other approaches.

Among papers considering some theoretical sides of the problem we mention [3]. As a paper devoted to the fibre orientations maximizing shear buckling load, with account of bending-twisting coupling, we indicate [4], containing important numerical results. The more comprehensive review may be found in [5].

The present paper is devoted to peculiarities of anisotropy/orthotropy optimization of composite plates under buckling conditions, in particular, under shear buckling conditions (in the latter case it is known, that the bending-twisting coupling is important for the loss of structural stability). In the Section 2 we indicate some theoretical features of the composite plate buckling problem. Then in the Section 3 we consider the role of bending-twisting coupling in optimality conditions and their solutions. After that, in the Section 4 we discuss a numerical composite plate example, illustrating theoretical analysis of the previous Sections. We consider also the ways of using the theoretical results in designing real structures against buckling.

2 Theoretical backgrounds

We consider the rectangular flat thin composite plate shown in the Figure 1.

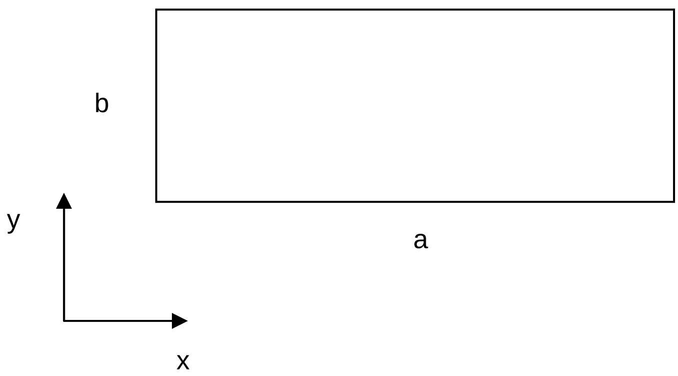

Figure 1. Rectangular plate.

The coordinate axes *x, y* are parallel to lateral sides of the plate, with the *x* axis being parallel to the longer side. The origin is located at the left bottom corner of the plate. In general, the plate is loaded by compression in two directions and shear. The signs of loads are the following. The flows N_x, N_y are positive in tension, the shear flow N_{xy} is positive when it decreases the 90° angle between the lateral sides at the origin. The flows may vary along edges and are in equilibrium. The Composite Lamination Plate Theory (CLPT) is used for description of plate deflections (see [6]). The plate is made of fiber-reinforced tape, with the lay-up being symmetric. The boundary conditions considered are the simple support and/or clamping. In the present paper we follow the known Bryan energy approach.
The equation for the plate deflection *w* is written as follows:

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}-$$
$$-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}-2N_{xy}\frac{\partial^2 w}{\partial x\partial y}=0 \qquad (1)$$

where $D_{ij}, i=1,2,6; j=1,2,6,$ are the elements of the bending stiffness matrix *D*, coupling the bending/twisting moments and various second derivatives of the deflection *w* with respect to *x* and *y*.

The kinematic variational principle used in the paper is written in the form of stationarity of the following ratio (under the same boundary conditions as before):

$$\delta\left[\frac{\tilde{\Pi}}{W^{(0)}}\right]=0 \tag{2}$$

where

$$\begin{aligned}\tilde{\Pi}=\iint dS\Big(&\frac{1}{2}D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2+D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{2}D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2+\\&+2D_{16}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x\partial y}+2D_{26}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 w}{\partial x\partial y}+2D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\Big)\end{aligned} \tag{3}$$

$$W^{(0)}=-\iint dS\left[\frac{1}{2}N_x^{(0)}\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}N_y^{(0)}\left(\frac{\partial w}{\partial y}\right)^2+N_{xy}^{(0)}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] \tag{4}$$

The eigen value, corresponding to the solution w of the considered buckling problem, may be calculated as the Rayleigh ratio:

$$\lambda=\frac{\tilde{\Pi}}{W^{(0)}} \tag{5}$$

The complementary and the mixed variational formulations are also indicated. It is shown how the buckling eigen value may be calculated using the complementary variational principle.

The reciprocity theorem for buckling states is proved, namely, the work performed by the buckling-induced forces of the state $'$ on deflections of the state $''$, is equal to the work performed by the forces of the state $''$ on deflections of the state $'$. The theorem is an analog of the known Betti theorem of structural mechanics in application to plate buckling. The theorem may be useful for checking the solutions and benchmarking.

The bending-twisting terms in (1) may be considered as small perturbations of the equation without the terms, as the values of D_{16}, D_{26} are, as a rule, not greater than 10-15% of D_{11}. Using the small perturbation approach, the following estimation of the first-order correction value $\Delta\lambda$ for the eigen value λ is obtained:

$$\Delta\lambda=-\frac{\iint dS\left[2D_{16}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial x\partial y}+2D_{26}\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w_0}{\partial x\partial y}\right]}{\iint dS\left[\frac{1}{2}N_x^{(0)}\left(\frac{\partial w_0}{\partial x}\right)^2+\frac{1}{2}N_y^{(0)}\left(\frac{\partial w_0}{\partial y}\right)^2+N_{xy}^{(0)}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]} \tag{6}$$

where w_0 is the solution of the eigen value problem with $D_{16}=D_{26}=0$.

3 The role of bending-twisting coupling in layer/laminate orientation optimization

3.1 Orthotropic case

We suppose the plate to be orthotropic and consider the role of the bending-twisting coupling in optimal choice of the plate orthotropy orientation. Saying “optimal” we mean maximizing the (lowest) buckling eigen value.

The orthotropic plate model is a model which allows describing some important features of the composite plate, taking into account the bending-twisting coupling generated by rotation of the orthotropy axes. That is why at first we consider an auxiliary optimization problem for an in-plane loaded orthotropic plate, with orthotropy axes orientation at every point being determined from optimization (maximization) of the lowest buckling eigen value. The boundary conditions considered are the same as in the previous Section. It is assumed that the eigen values are simple. It is known, that the simple eigen value optimality consideration is a part of the multiple eigen value optimality one (the latter takes into account the possible non-differentiability of the minimal eigen value to be maximized).

We assume that the plate bending stiffness matrix D in local material axes is the same at every point of the plate. The orientation of the local material axes is determined by the angle θ (see Figure 2, where the angle is negative) between the x axis and the local material axis corresponding to D_{11} (the strongest direction).

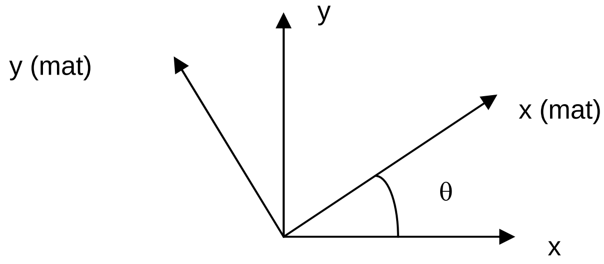

Figure 2. Local material axes orientation.

Resulting from the optimization problem, the angle *θ(x,y)* as a smooth function must be determined.

The deflections of the buckling state of the plate for every function *θ(x,y)* may be determined by the kinematic variational principle (2). According to the regular way of obtaining the optimality conditions in such optimization problems, the buckling eigen value given by the Rayleigh ratio is maximized. The necessary optimality condition is (for all points within the plate):

$$\bar{k}^T \frac{dD}{d\theta} \bar{k} = 0 \tag{7}$$

where $\bar{k}$ is the vector composed of $\left(-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x \partial y}\right)$. The condition may be considered as calculated in the coordinate system corresponding to the principal curvature lines of the surface *w(x,y)* to the *x, y* plane. We denote the principal curvature values as k_1 and k_2, where $k_1 \geq k_2$. Also in the case the angle $\omega = \theta - \psi$ is the angle between material axis 1 (corresponding to D_{11}^{mat}) and the principal curvature direction 1 (corresponding to k_1), ψ is the angle between x axis and k_1 direction.

Finally (7) leads to the following relation

$$D_{16}^{pr.cur.} k_1^2 - D_{26}^{pr.cur.} k_2^2 - \left(D_{16}^{pr.cur.} - D_{26}^{pr.cur.}\right) k_1 k_2 = 0 \tag{8}$$

where the elements of the D matrix correspond to the coordinate system of the principal curvature axes, as indicated by the superscript *pr.cur.* The relation (8) combines the conditions:

$$k_1 = k_2 \tag{9}$$

$$D_{16}^{pr.cur.} = D_{26}^{pr.cur.} \tag{10}$$

$$D_{16}^{pr.cur.} k_1 + D_{26}^{pr.cur.} k_2 = 0 \tag{11}$$

(8) may be also written as:

$$(k_1 - k_2) M_{xy}^{pr.cur.} = 0 \tag{12}$$

The obtained expressions (8)-(12) clearly demonstrate the role of the $D_{16}^{pr.cur.}, D_{16}^{pr.cur.}$ quantities and of the principal curvature values. As we see, the equality to zero of the twisting moment in non-umbilical points is a fundamental feature of the optimal solution.
Further we consider the optimal lay-up orientation of the composite orthotropic plate against buckling. The only parameter to be varied is the orientation angle of the whole plate. The solution of the problem is, to some extent, “an average” of the result of the above optimization problem. The derivation of the necessary optimality condition is similar to the locally orthotropic plate case. The condition is written as

$$\iint dS \left[\bar{k}^T \frac{dD}{d\theta} \bar{k} \right] = 0 \tag{13}$$

where the angle θ is considered as being of the same value for the whole plate, with the value being a purpose of the optimization.

Making necessary transformations, we obtain:

$$\iint dS \left[D_{16}^{pr.cur.} k_1^2 - D_{26}^{pr.cur.} k_2^2 - \left(D_{16}^{pr.cur.} - D_{26}^{pr.cur.} \right) k_1 k_2 \right] = 0 \tag{14}$$

or, in another form:

$$\iint dS (k_1 - k_2) M_{xy}^{pr.cur.} = 0 \tag{15}$$

which physically means, that the twisting moment in principal curvature axes must have some “balanced” and close to zero value. The multiplier $k_1 - k_2$ penalizes the deviation of the moment from zero: the more is the difference in principal curvature values, the greater is the input into the integral in (15).
Concluding the discussion of the two optimization problems and corresponding optimality conditions, one may say, that for the structure optimal in the first problem there is no twisting moment in principal curvature axes (except umbilical points), or, in other words, the moment tensor and the tensor of curvatures are co-axial. For the structure optimal in the second problem the twisting moment is not necessarily equal to zero in the structure, but the averaged value of the moment with the multiplier $k_1 - k_2$ is equal to zero.

3.2 Anisotropic case

In the sub-Section we consider the role of the bending-twisting coupling in optimal choice of the plate anisotropy (layer orientation distribution) for maximizing the (lowest) eigen value. The eigen value is supposed to be simple. We use lamination parameter approach [6].

As a first (auxiliary) task we consider determination of layer orientation angles θ_i, i=1,…,N, as smooth functions of (x,y), where 2*N is the total number of layers (we do not consider separately the case of odd number of layers in the plate lay-up because of non-significant complication of the analysis).

The optimality conditions like (7) are also valid for the case, namely:

$$\bar{k}^T \frac{dD}{d\theta_i}\bar{k} = 0 \qquad i=1,...,N \tag{16}$$

The solutions of (16) for i=1,...,N are:

$$k_1 = k_2 \tag{17}$$

$$\sin 2(\theta_i - \psi) = 0 \tag{18}$$

$$\cos 2(\theta_i - \psi) = -\frac{U_2(k_1+k_2)}{U_3(k_1-k_2)} = -\frac{(k_1+k_2)}{(k_1-k_2)}\frac{(Q_{11}-Q_{22})}{\left(Q_{11}+Q_{22}-2(Q_{12}+2Q_{66})\right)} \tag{19}$$

where $U_i, i=1,...,4,$ are some layer elastic parameters [6]. The relation (17) corresponds to the umbilical points. The relation (18) corresponds to $\theta_i - \psi$ =0 and $\pm \pi/2$. The quantities $Q_{11}, Q_{12}, Q_{22}, Q_{66}$ are the elements of the layer stiffness matrix (in local coordinates), depending on layer elastic properties $E_1, E_2, \nu_{12}, \nu_{21}, G_{12}$ only. There are only two solutions of (19), namely

$$\theta_i - \psi = \pm \vartheta_0 \tag{20}$$

where $\vartheta_0 \in \left[0, \pi/2\right]$ and ϑ_0 depends on a point at the plate via k_1, k_2.

Making summation of the derivatives of the stiffness matrix elements, we obtain the above relation (8), and, hence, (12). Thus, the relations (8) and (12) are the consequence of the necessary optimality conditions (16). It is clear, that at the non-umbilical points the twisting moment in principal curvature axes is equal to zero.

The second task, to be considered, differs from the first one of the sub-Section in the layer orientation angle distributions, namely, the angles have the values permanent for the whole plate. As above, we maximize the (lowest) buckling eigen value, with the orientation angles being the variables.

The optimality conditions in the case are:

$$\iint dS\left[\bar{k}^T \frac{dD}{d\theta_i}\bar{k}\right] = 0 \qquad i=1,...,N \tag{21}$$

Calculating (21) in the principal curvature axes leads to the above-obtained relation (14), and, hence, to (15). Thus, the relations (14) and (15) are the consequence of the necessary

optimality conditions (21). As a combination of the relations (21), the conditions (14) or (15) are also the variants of necessary optimality condition, allowing making the same conclusions about bending-twisting coupling as ones made in the above sub-Section.
Ending the Section 3, we conclude, that the bending-twisting coupling terms are presented by the same expressions in the considered necessary optimality conditions for both orthotropic and anisotropic cases.

4 Numerical examples

The example problems of a thin long simply supported composite plate, loaded by shear, are considered.
For the first problem the lay-up is symmetric and, if it is not said different, orthotropic $\left[\left[\theta/(\theta+90°)\right]_4\right]_s$, where θ is the lay-up orientation angle to be optimized. The angle corresponds to the orientation of the strongest material axis. The plate consists of 16 unidirectional layers of t300/5208 CFRP material and has the total thickness of 2 mm. The plate dimensions are 200 by 1000 mm. The reference in-plane shear flow applied is equal to 50 N/mm. The numerical analysis is performed by FEM.
The buckling analysis performed for typical shear-targeted lay-up +/-45°, results in λ=1.30; -1.67 (sometimes we will indicate the eigen value with the negative sign, which means the negative shear loading direction).
Analysis performed using small perturbation approach (6) combined with energy approach [7] for the specially orthotropic material gives λ=+/-1.49-0.20. As we see the approach (6) provides results very close to ones, calculated directly by FEM.
The Figure 3 demonstrates the shear buckling eigen values for various orientation angle θ. As we see, there are two maximums: for lay-ups 28°/118° (λ=1.47) and 60°/150° (λ=-1.93). The second one is the global one for the considered lay-ups. The lowest buckling modes for the maximums look like inclined (about 55°-60° to long side direction) waves, with the distance between the zero-deflection lines being approximately equal to 1.3*b*.

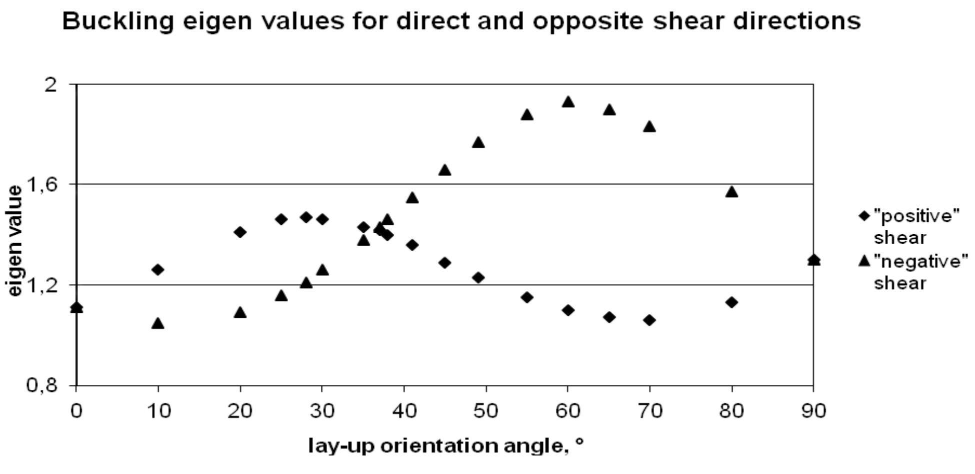

Figure 3. Dependence of the buckling eigen values from the lay-up orientation angle θ.

It is seen from the Figure 3, that the lay-up rotation up to 38° leads to equalizing the eigen vale at λ=+/-1.43 for both shear directions. For comparison, if the clamped boundary conditions are used, then the equalizing occurs at 35°.
Direct check of the unidirectional (UD) laminate 60° (the configuration was founded numerically in [4]), leads to λ= 0.42 and -2.82. The positive shear has very low buckling level, which is inacceptable.

For the laminate with the lay-up +/-60° the lowest buckling values are 1.56 and -2.00, with the distance between the zero-deflection lines being approximately equal to 1.1*b*. The waves are more inclined (the angle is closer to 60°).
Finally the best solution found is the lay-up $\left[\left[74°/134°\right]_4\right]_S$ with the lowest λ=~ +/-1.70 (the layer 74° is the outermost one). The waves are inclined with ~60° to the long side direction, and the distance between the zero-deflection lines is approximately equal to *b*. The benefit in buckling level comparing with the +/-45° lay-up is 30%. As we see, there is a promising plate weight saving potential. The Figure 4 demonstrates the optimal lowest buckling mode obtained.

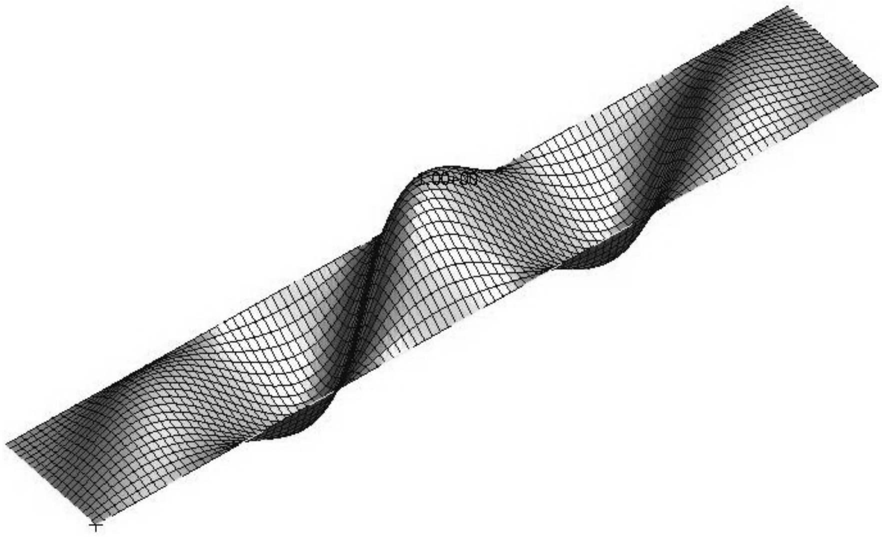

Figure 4. The lowest buckling mode for the lay-up 74°/134°.

In all cases, considered in the Section, the lowest λ corresponding to the maximums are doubled in the vicinity of the solution.

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