DYNAMICS OF AXIALLY LOADED AND PARTIALLY INTERACTING COMPOSITE BEAMS

S. Grundberg[1*], O. A. B. Hassan[1], U. A. Girhammar[2]

[1]*Dept. of Applied Physics and Electronics, Umeå University, SE-901 87 Umeå, Sweden*
[2]*Division of Structural and Construction Engineering-Timber Structures, Luleå University of Technology, SE-971 87 Luleå, Sweden*
[*]*staffan.grundberg@tfe.umu.se*

Keywords: Dynamics of composite beams, interlayer slip, natural frequency, axial loading, boundary conditions

Abstract
The aim of this paper is to study the dynamics of partially interacting composite beams subjected to axial loads. The eigenfrequencies are derived for the simply supported composite beam subjected to an axial force. Measurements of the fundamental frequencies of a simply supported composite beam consisting of three layers connected by shear connectors are carried out considering different values of the applied axial force. The agreement between the theoretical and experimental results is discussed. It is also shown that the application of the axial force introduces frictional moments at the boundaries, which make the experimental boundary conditions deviate somewhat from those of an ideal simply supported beam.

1 Introduction

Composite structures with partial interaction are frequent in the civil engineering and construction industry. As an example, floor and wall elements in buildings are often composite structures, e.g. of concrete and wood interacting partially through shear connectors. To obtain full advantage of composite beams, the layers of the composite beams should act compositely. Composite action enhances the static and dynamic behaviour of the beam significantly and allows long clear spans with a slim floor depth giving many benefits for multi-storey building design.

This study is aimed at studying the dynamic behaviour of partially composite beams subjected to static axial forces. An important aspect of the dynamic analysis of composite beams is the determination of their natural frequencies, damping, deformations and mode shapes [1]. This is important because composite beam structures often operate in different environmental conditions and are often exposed to a variety of dynamic excitations such as axial excitation. These composite structures can be subject both to static and dynamic loads. The statics of composite structures with partial interaction have been considered by e.g. Girhammar [2]. The dynamics of composite structures have been studied e.g. by Henghold [3] and Girhammar et al. [4], where the equations of motion were derived from Hamilton's principle. Their results were derived for the case of no axial load. Theoretical results for the eigenfrequencies of axially loaded beams have been presented by Wu et al. [5].

Coulter and Miller [6] studied free vibration of elastic Euler-Bernoulli beams subjected to non-uniform axial forces and the buckling of elastic columns using beam finite elements. Bokaian

[7, 8] presented a set of transcendental equations to derive the natural frequency and the load ratio, for several combinations of boundary conditions including simply supported, sliding, free or clamped ends. A uniform beam under constant axial compressive force and tensile axial force was considered. Guédé and Elishakoff [9] derived closed-form solutions for the fundamental natural frequencies of inhomogeneous vibrating beams under axially distributed loading. Several sets of boundary conditions were considered. It was shown that the natural frequency vanishes when the intensity of the axially distributed loading equalled the critical buckling value. Mok and Murray [10] presented theoretical and experimental results for the vibrations of inhomogeneous columns subjected to axial loading. Yeh and Liu [11] investigated the vibrations of inhomogeneous columns subjected to axial loading using the Galerkin method for a uniform beam column with rotational and translational restraints. Williams and Banerjee [12] investigated the free vibrations of axially loaded beams with linear or parabolic taper for various sets of boundary conditions. Banerjee and Williams [13] determined the first five natural frequencies of axially loaded tapered columns for eleven combinations of boundary conditions, within the Bernouilli-Euler and the Bresse-Timoshenko beam theories. Gottlieb [14] derived seven different classes of inhomogeneous Bernouilli-Euler beams of continuously varying material density and flexural stiffness. Other related studies are those by Gajewski [15], Datta and Nagraj [16], Shaker [17], Glück [18]. Liu et al. [19] investigated the coupled axial-torsional vibration of pre-twisted beams. The equations of motion governing the extension, torsion, and cross-sectional warping of pre-twisted beams were derived from Hamilton's principle, and the common assumptions used to simplify the equations were carefully examined through scaling analysis. Chen and Ho [20] studied the problem of transverse vibration of rotating twisted Timoshenko beams under axial loading using the differential transform method to obtain natural frequencies and mode shapes. Naguleswaran [21] derived an approximate solution to the equations of transverse vibration of the uniform Euler-Bernoulli beam under linearly varying axial force.

2 Equations of motion

The equations of motion are obtained by first finding the Lagrangian, and then applying Hamilton's principle [22]. In the case of three identical layers, see Fig. 1, it is most convenient to eliminate the axial force in the middle layer, and the equations of motion reduce to

$$-EI_0\frac{\partial^4 w}{\partial x^4} - \Delta z\frac{\partial^2 \Delta N}{\partial x^2} + F\frac{\partial^2 w}{\partial x^2} = m\frac{\partial^2 w}{\partial t^2} \tag{1}$$

$$\frac{\partial^2 \Delta N}{\partial x^2} - K\left(\frac{\Delta N}{EA} - 2\Delta z\frac{\partial^2 w}{\partial x^2}\right) = 0, \tag{2}$$

where w is the lateral displacement according to Fig. 1 and $\Delta N = N_1 - N_3$ the difference between the axial forces in the outermost layers. The sum of the bending stiffnesses of the three layers is denoted by EI_0 and the slip modulus, i.e. the proportionality constant between the interlayer force and the interlayer slip, by K. The axial force is denoted by F. The mass per unit length is denoted by m and the distance between the centroids of adjacent layers are denoted by Δz. Eliminating ΔN yields

$$\frac{\partial^6 w}{\partial x^6} - \left(\alpha^2 + \frac{F}{EI_0}\right)\frac{\partial^4 w}{\partial x^4} + \frac{F\alpha^2}{EI_\infty}\frac{\partial^2 w}{\partial x^2} - \frac{1}{EI_0}\frac{\partial^2 q}{\partial x^2} + \frac{\alpha^2}{EI_\infty}q = \frac{m\alpha^2}{EI_\infty}\frac{\partial^2 w}{\partial t^2} - \frac{m}{EI_0}\frac{\partial^4 w}{\partial x^2 \partial t^2}, \tag{3}$$

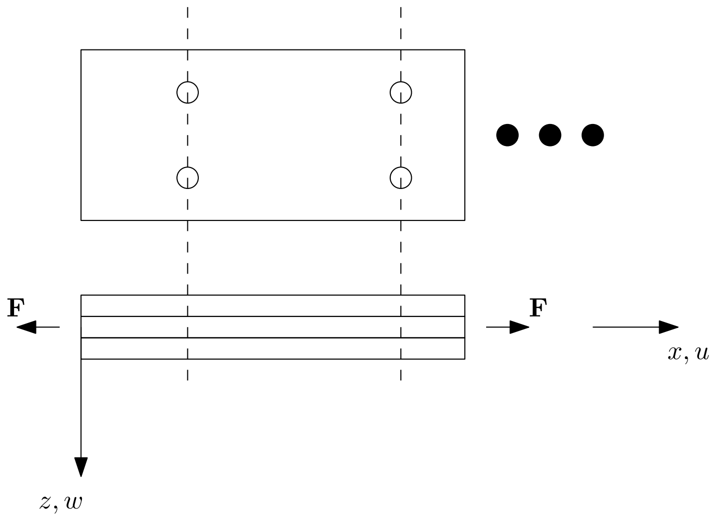

Figure 1. The composite beam consisted of three layers of aluminium connected by plugs of the material polyoxymethylene(POM). The individual layers had a width of 80 mm and height 5 mm. The overall length of the beam is 2 m. There were maximally 19 pairs of plugs at a distance of 10 cm.

where the bending stiffness of the fully composite beam is

$$EI_\infty = EI_0 \left(1 + \frac{2EA(\Delta z)^2}{EI_0}\right) \tag{4}$$

and

$$\alpha^2 = K \left(\frac{1}{EA} + \frac{2(\Delta z)^2}{EI_0}\right). \tag{5}$$

At a pinned end, there is no moment ($M_B = 0$), no lateral displacement ($w = 0$) and a prescribed axial force ($\Delta N = 0$). This results in the following set of boundary conditions for a pinned end

$$\frac{\partial^2 w}{\partial x^2} = 0 \tag{6}$$

$$w = 0 \tag{7}$$

$$\frac{\partial^4 w}{\partial x^4} - \frac{q}{EI_0} = 0. \tag{8}$$

2.1 Dynamic solution for a simply supported beam

Consider a beam with both ends pinned, one at $x = 0$ and the other at $x = L$. Eq. (3) is solved by the method of separation of variables [22, 23], i.e. the displacement is written as

$$w(x,t) = \phi(x) f(t) \tag{9}$$

yielding the equations

$$\frac{d^6\phi}{dx^6} - \left(\alpha^2 + \frac{F}{EI_0}\right)\frac{d^4\phi}{dx^4} + \frac{F\alpha^2}{EI_\infty}\frac{d^2\phi}{dx^2} - \frac{m\omega^2}{EI_0}\frac{d^2\phi}{dx^2} + \alpha^2\frac{m\omega^2}{EI_\infty}\phi = 0 \tag{10}$$

$$\frac{d^2 f}{dt^2} + \omega^2 f = 0. \tag{11}$$

b (mm)	**h (mm)**	**L (m)**
80	5	2.00
$\mathbf{EI_0}$ **($\mathbf{Nm^2}$)**	$\mathbf{EI_\infty}$ **($\mathbf{Nm^2}$)**	**m (kg/m)**
175	1575	3.24
$\mathbf{k_{1,1}}$ **($\mathbf{m^{-1}}$)**	$\mathbf{f_{\infty,1}}$ **(Hz)**	$\mathbf{P_{\infty,cr,1}}$ **(kN)**
1.57	8.65	3.89

Table 1. Geometric parameters(width, height and length), the bending stiffness in the cases of no interaction as well as full interaction and mass per unit length. The table also contains the theoretical values of the wave number of the fundamental mode, the fundamental frequency, and the critical force. The values are calculated from the values $E = 70$ GPa and $\rho = 2700$ kg/m^3 for aluminium.

A solution of the form $\phi(x) = \exp(kx)$ yields the characteristic equation

$$k^6 - \left(\alpha^2 + \frac{F}{EI_0}\right)k^4 - \left(\frac{m\omega^2}{EI_0} - \frac{F\alpha^2}{EI_\infty}\right)k^2 + \alpha^2\frac{m\omega^2}{EI_\infty} = 0, \tag{12}$$

and application of the boundary conditions for pinned ends both at $x = 0$ and $x = L$ yields the eigenfrequencies

$$\omega_n^2 = \omega_{\infty,n}^2\left[1 + \frac{F}{P_{\infty,\mathrm{cr},n}} - \left(1 + \frac{1 + (\alpha/k_{1,n})^2}{(EI_\infty/EI_0) - 1}\right)^{-1}\right], \tag{13}$$

where the wave number is given by

$$k_{1,n} = \frac{n\pi}{L}, \tag{14}$$

and the eigenfrequencies of a fully composite beam are given by

$$\omega_{\infty,n}^2 = \frac{EI_\infty k_{1,n}^4}{m}. \tag{15}$$

The critical compressive force of the fully interacting composite beam is denoted by

$$P_{\infty,\mathrm{cr},n} = EI_\infty k_{1,n}^2. \tag{16}$$

3 Method and equipment

The experiments were performed on a composite beam consisting of three layers of aluminium having the length 2 m and a rectangular cross section of width 80 mm and height 5 mm. The aluminium layers were connected by 19 plugs of the material polyoxymethylene (POM) having the diameter 8 mm, see Fig. 1. The different parameters of the beam and fundamental frequency and critical force in the case of full interaction are summarised in table 1.

The composite beam was mounted in a frame exerting the axial compressive force on the beam, see Fig. 2. The frame consisted essentially of steel plates on each side of the composite beam. The plates were connected by two threaded bars and the axial force was adjusted by changing the angular position of the nuts supporting the plates. The force was measured by measuring the compression of two sets of cup springs, where each set consisted of 40 cup springs, see Fig. 2. The whole assembly of cup springs had an effective spring constant of 460 N/mm. The supports were designed to allow free rotation of the beam while also transferring the axial

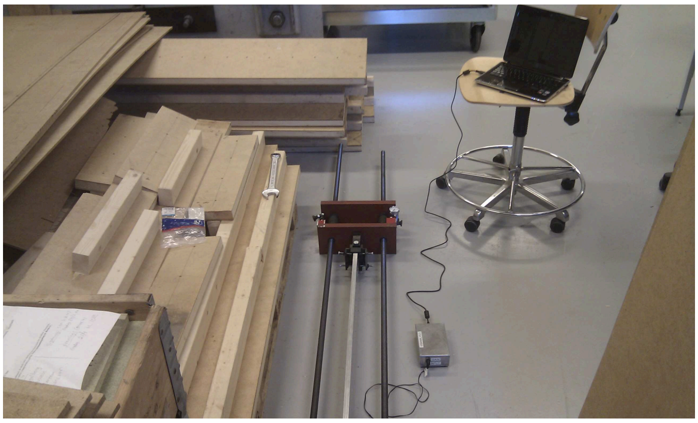

Figure 2. The accelerometer is connected to a signal conditioning device, which is connected to a laptop computer for data retrieval.

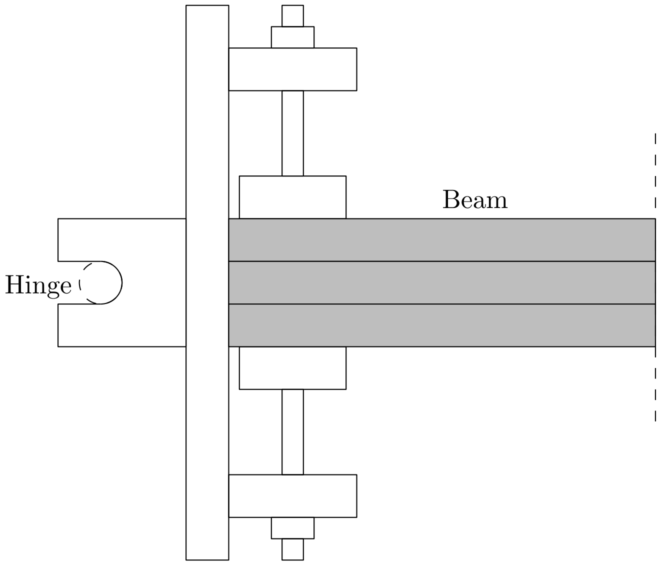

Figure 3. The figure shows the support at one end of the beam. The beam is clamped between two plates and the whole support device is hinged at the end.

force to the beam, see Fig. 3. At the support, the beam is clamped between two plates and the whole support device is hinged at the end, see Fig. 3. The supports are sources of both additional inertia and friction at the boundaries. The acceleration of the beam was measured by a capacitive accelerometer (LIS3L06AL MEMS inertial sensor) from ST Microelectronics connected to a signal conditioning device designed and built by the Mechanics Division in the Department of Applied Physics and Electronics at Umeå University, see Fig. 2. The accelerometer had a frequency range from DC up to 1.5 kHz. The accelerometer signal conditioning device was connected to a laptop computer running the software Audacity for storing the recorded signal in wav-format, see Fig. 2. The slip modulus K or, equivalently, the parameter αL (see Eq. (5)) of the partial interaction of the beam was determined from measurement of the steady state deflection of the beam under a static load at mid-span. The deflection was measured using a dial gauge.

The vibration experiments were first performed on a beam connected by 19 plugs without axial force and then again with an axial force applied. The accelerometer was mounted at mid-span of the beam in order to detect the fundamental mode of the beam. The vibrations of the beam were

F (kN)	f_1 (Hz) Theory		f_1 (Hz) Experiment	Damping ratio ζ Experiment
	S.S.	C.C.		
0	7.2	19.6	9.2	0.09
-1.6	4.6	12.1	10.9	0.02

Table 2. Comparison of theoretical and experimental results. S.S and C.C are abbreviations of simply supported and clamped-clamped, respectively.

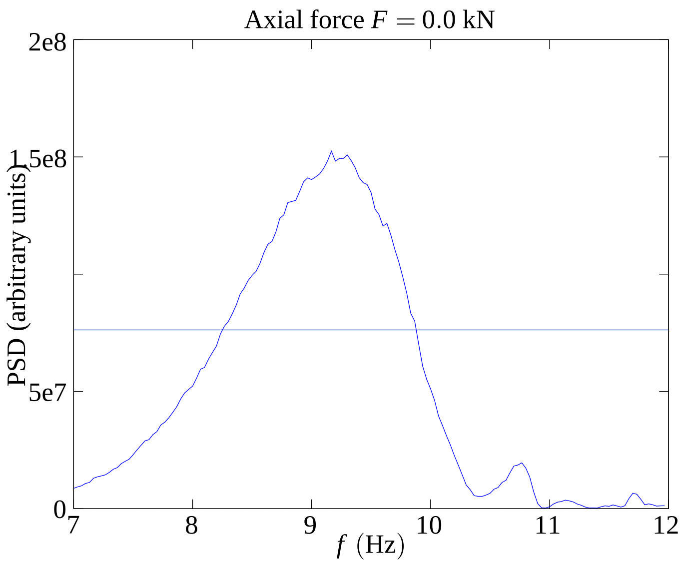

Figure 4. Acceleration spectrum in the case of no axial force. The spectrum has a maximum at around 9.2 Hz.

excited by a gentle manual push and the oscillations were the recorded during a time interval lasting around 40 s.

4 Results

The deflection measurement yielded an interaction parameter $\alpha L = 13$. The frequency of the fundamental mode in the absence of axial force was measured to be 9.2 Hz and the corresponding modal damping ratio was 0.09, see Fig. 4. When an compressive axial force $F = -1.6$ kN was applied, then the frequency of the fundamental mode increased to 10.9 Hz and the damping ratio decreased to 0.02, see Fig 5. The theoretical value of the fundamental frequency was 7.2 Hz in the case of zero force, and 4.6 Hz in the case of an compressive axial force $F = -1.6$ kN. The corresponding theoretical values for the clamped-clamped beam were 19.6 and 12.1 Hz, respectively. The results are summarized in table 2.

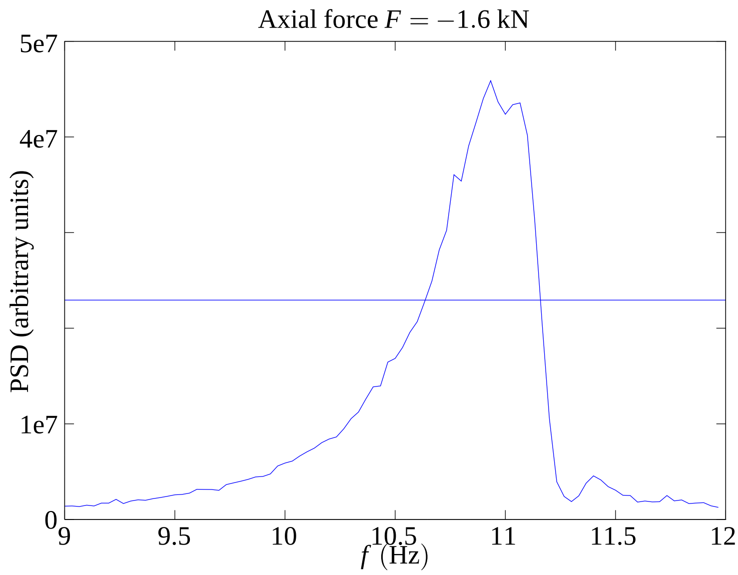

Figure 5. Acceleration spectrum in the case of an axial compressive force of 1.6 kN. The spectrum has a maximum at around 10.9 Hz.

5 Conclusions

The experimental result that the fundamental frequency of the beam increased as the axial force increased was in contradiction to the theoretical result for the simply supported beam. A possible explanation is that the supports impede the motion of the beam at the boundaries due to both friction and inertia. The boundary conditions can therefore no longer be considered as pinned. Instead, the actual boundary conditions are somewhere in between the pinned and clamped cases. Observe that the experimental fundamental frequency is somewhat higher than the theoretical value for the simply supported beam in the case of zero axial, but significantly lower than the theoretical value for the clamped-clamped beam. When the axial force is applied, then the experimental fundamental frequency instead approaches the theoretical value for the clamped-clamped beam from below. This indicates that the boundary conditions become closer to clamped than pinned as the axial force is applied. One possible explanation could be that the friction at the boundaries increases as the axial force is increased. A more detailed analysis of the boundary conditions with respect to friction and inertia is needed (see e.g. [4] with respect to boundary effects).

Further work must be spent on designing supports with lower friction in order to get a better agreement with the theoretical results for the fundamental frequency of an axially loaded simply supported composite beam.

Acknowledgement

The authors wish to express their sincere appreciation for the financial support from The European Union's Structural Funds — The Regional Fund.

References

[1] Hassan O.A.B. *Building acoustics and vibration*. World scientific publications (2009).

[2] Girhammar U.A., Pan D.H. Exact static analysis of partially composite beams and beam-columns. *International Journal of Mechanical Sciences*, **49**, pp. 239–255 (2007).

[3] Henghold W.M. *Layered beam vibrations including slip*. Ph.D. thesis, Colorado State University (1972).

[4] Girhammar U.A., Pan D.H., Gustafsson A. Exact dynamic analysis of composite beams with partial interaction. *International Journal of Mechanical Sciences*, **51**, pp. 565–582 (2009).

[5] Wu Y.F., Xu R., Chen W. Free vibrations of the partial-interaction composite members with axial force. *Journal of Sound and Vibration*, **299**, pp. 1074–1093 (2007).

[6] Coulter B.A., Miller R.E. Vibration and buckling of beam-columns subjected to non-uniform axial loads. *International Journal for Numerical Methods in Engineering*, **23(9)**, pp. 1739–1755 (1986).

[7] Bokaian A. Natural frequencies of beams under compressive axial loads. *Journal of Sound and Vibration*, **126**, pp. 49–65 (1988).

[8] Bokaian A. Natural frequencies of beams under tensile axial loads. *Journal of Sound and Vibration*, **142**, pp. 481–498 (1990).

[9] Guédé Z., Eliashakoff I. Apparently first closed form solutions for inhomogeneous vibrating beams under axial loading. *Proc. R. Soc. Lond. A*, **457**, pp. 623–649 (2001).

[10] Mok C.H., Jr E.A.M. Free vibrations of a slender bar with nonuniform characteristics. *J. Acoust. Soc. Am.*, **40**, pp. 385–389 (1965).

[11] Yeh F.H., Liu W.H. Free vibrations of nonuniform beams with rotational and translational restraints and subjected to an axial force. *J. Acoust Soc. Am.*, **85**, pp. 1368–1371 (2001).

[12] Banerjee J.R., Williams F.W. Further flexural vibration curves for axially loaded beams with linear and parabolic taper. *Journal of Sound and Vibration*, **102**, pp. 315–327 (1986).

[13] Williams F.W., Banerjee J.R. Flexural vibrations of axially loaded beams with linear or parabolic taper. *Journal of Sound and Vibration*, **99**, pp. 121–138 (1986).

[14] Gottlieb W. Isospectral Euler-Bernoulli beams with continuous density and rigidity functions. *Proc. R. Soc. Lond. A*, **413**, pp. 235–250 (1987).

[15] Gajewski A. Note on unimodal and bimodal optimal design of vibrationg compressed columns. *Int. J. Mech. Sci.*, **23**, pp. 11–16 (1981).

[16] Datta P.K., Nagraj C.S. Dynamic instability behaviour of tapered bars with flawssupported on an elastic foundation. *J. Sound Vib.*, **131**, pp. 227–229 (1986).

[17] Shaker F.J. Effect of axial load on mode shape and frequencies of beams. Technical Report Report NASATN-8109, NASA Lewis Research Center (1975).

[18] Glück J. Vibration frequencies of an elastically supported cantilever column with variable cross-section and distributed axial load. *Earthquake Engng. Struct. Dynam.*, **1**, pp. 371–376 (1973).

[19] Liu K.C., Friend J., Yeo L. The axial-torsional vibration of pretwisted beams. *Journal of Sound and Vibration*, **321**, pp. 115–136 (2009).

[20] Chen C.K., Ho S.H. Transverse vibration of rotating twisted Timoshenko beams under axial loading using differential transform. *International Journal of Mechanical Sciences*, **41**, pp. 1339–1356 (1999).

[21] Naguleswaran S. Transverse vibration of an uniform Euler Bernoulli beam under linearly varying axial force. *Journal of Sound and Vibration*, **275**, pp. 47–57 (2004).

[22] Fetter A.L., Walecka J.D. *Theoretical mechanics of particles and continua*. McGraw-Hill (1980).

[23] Volterra E., Zachmanoglou E.C. *Dynamics of vibrations*. Clarles E. Merril Books, Columbus, Ohio (1965).