EXPLICIT SECULAR EQUATIONS AND FORMULAS FOR THE VELOCITY OF RAYLEIGH WAVES IN A DIRECTIONAL FIBER-REINFORCED COMPOSITE

Pham Chi Vinh[1*] and Nguyen Thi Khanh Linh[2]

[1] *Faculty of Mathematics, Mechanics and Informatics, Hanoi University of Science, 334, Nguyen Trai Str., Thanh Xuan, Hanoi,Vietnam*

[2] *Department of Engineering Mechanics, Water Resources University of Vietnam, 175 Tay Son Str., Hanoi, Vietnam*

* pcvinh@vnu.edu.vn

Keywords: Rayleigh waves, Formula for the velocity, Explicit secular equation, Directional fiber-reinforced composite.

Abstract

In this paper, first we derive an explicit secular equation of Rayleigh waves whose propagation direction being parallel to the fiber direction. Then, an exact formula for the velocity is derived and two good approximations for it are created. Secondly, for the case of the propagation direction of Rayleigh waves being oblique to the fiber direction, an explicit secular equation has been obtained. The obtained explicit secular equations and formulas for the velocity will be very useful for analyzing the effect of the material properties and the orientation of the fiber direction on the Rayleigh wave propagation, especially they will be powerful tools for solving the inverse problem: determining the material parameters from the measured values of the velocity.

1 Introduction

The propagation of a Rayleigh-edge wave in a principal material direction of a thin semi-infinite orthotropic panel, called a principal Rayleigh-edge wave, was investigated recently by Cerv [1] and Cerv et al. [2]. The panel may be a laminate, isotropic medium or slightly anisotropic crystallic medium on one hand or a crystallic structure with strong cubic orthotropy on the other hand. The authors derived the secular equations of the wave for both types of material, and they are not the same. The propagation of a non-principal Rayleigh-edge wave, the wave whose propagation direction does not coincide with the principal material axes, in a thin semi-infinite orthotropic panel was also investigated recently by Ohyoshi [3] and Cerv et al. [2]. However, an explicit secular equation of this wave has not been yet obtained in both investigations.

In this paper we provide a secular equation for the principal Rayleigh-edge waves that is the same for all orthotropic elastic materials and much more simple than the ones obtained by Cerv [1] and Cerv et al. [2]. An exact formula for the velocity of principal Rayleigh-edge waves are derived, and two approximate formulas for the velocity are established by using the best approximate second-order polynomials in the interval [0, 1] of the cubic power [4]. It is shown that they are good approximations. For non-principal Rayleigh-edge waves an explicit secular equation is obtained by using the method of first integrals [5, 6].

2 **Principal Rayleigh-edge waves in thin orthotropic media**

2.1 Secular equation

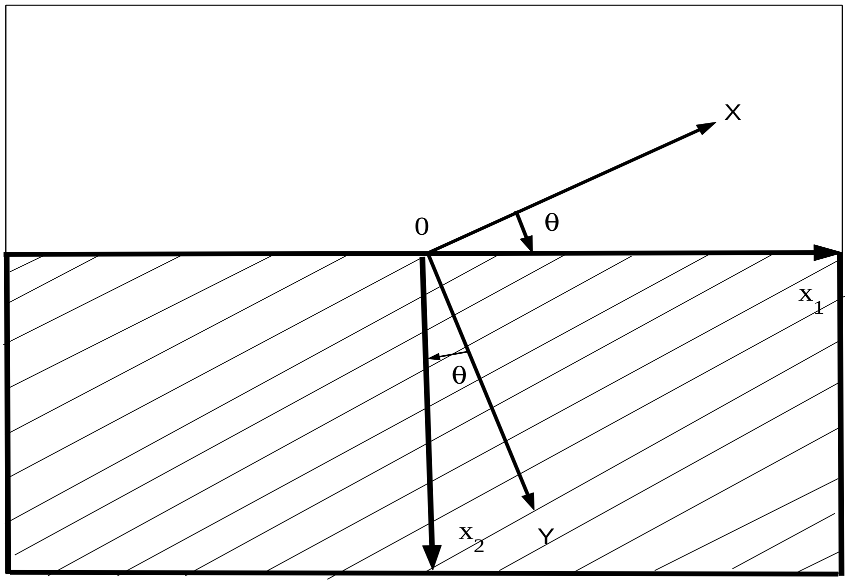

Figure 1: The thin orthotropic panel $x_2 \geq 0$ with principal material axes X, Y, Z, the Z-axis coincides with the x_3-axis and coordinate system (x_1, x_2) is the rotated one from (X, Y) by counter clockwise angle θ.

Consider a thin semi-infinite orthotropic medium (panel) occupying the half-space $x_2 \geq 0,$ its principal material axes are x_1-, x_2- and x_3-axis (Fig. 1 with $\theta = 0$) and it is in the state of plane stress:

$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \tag{1}$$

The components of the stress tensor $\sigma_{ij}, i, j = 1,2$ are related to the displacement gradients by the following equations:

$$\sigma_{11} = B_{11}u_{1,1} + B_{12}u_{2,2},\ \sigma_{22} = B_{12}u_{1,1} + B_{22}u_{2,2},\ \sigma_{12} = B_{66}(u_{1,2} + u_{2,1}) \tag{2}$$

where u_1, u_2 are displacement components, commas indicate differentiation with respect to partial variables x_k, B_{ij} are material (stiffness) coefficients which can be expressed in terms of the engineering constants (Young's and shear moduli, Poisson's ratios) as [2]:

$$B_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ B_{22} = \frac{E_1}{1-\nu_{12}\nu_{21}},\ B_{12} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ B_{66} = G_{12} \tag{3}$$

and satisfy the inequalities:

$$B_{kk} > 0,\ k = 1,2,6,\ B_{11}B_{22} - B_{12}^2 > 0 \tag{4}$$

which are necessary and sufficiently for the strain energy of the material to be positive define.

In the absence of body forces, equations of motion are:

$$\sigma_{11,1} + \sigma_{12,2} = \rho\ddot{u}_1,\ \sigma_{12,1} + \sigma_{22,2} = \rho\ddot{u}_2 \tag{5}$$

here, a superposed dot signifies the differentiation with respect to
the time t and ρ is the mass density. Substituting (2) into (5) yields (see also [1]):

$$\begin{aligned} B_{11}u_{1,11} + B_{66}u_{1,22} + (B_{12}+B_{66})u_{2,12} &= \rho\ddot{u}_1 \\ B_{66}u_{2,11} + B_{22}u_{2,22} + (B_{12}+B_{66})u_{1,12} &= \rho\ddot{u}_2 \end{aligned} \tag{6}$$

These equations are taken together with the traction-free condition:

$$\sigma_{2i} = 0,\ i = 1,2 \ \text{ on } \ x_2 = 0 \tag{7}$$

and the decay condition:

$$u_i = 0,\ \sigma_{2i} = 0,\ i = 1,2 \ \text{ on } \ x_2 = +\infty \tag{8}$$

We now consider the propagation of a principal Rayleigh-edge wave, travelling with the velocity c and the wave number *k* in the x_1-direction and exponentially decay in the x_2-direction. The wave displacement components are sought in the form:

$$u_i = U_i(y)\exp[kx_1 - ct],\ i = 1,2,\ y = kx_2 \tag{9}$$

Following the same procedure carried out in [7] one can see that:

$$U_1 = A_1\exp(-s_1 y) + A_2\exp(-s_2 y),\ U_2 = A_1 q_1\exp(-s_1 y) + A_2 q_2\exp(-s_2 y) \tag{10}$$

where s_1, s_2 are the roots of the equation:

$$B_{22}B_{66}s^4 + [(B_{12}+B_{66})^2 + B_{22}(\rho c^2 - B_{11}) + B_{66}(\rho c^2 - B_{66})]s^2 + (B_{11} - \rho c^2)(B_{66} - \rho c^2) = 0 \tag{11}$$

having positive real parts to ensure the decay condition (8), q_k $(k = 1,2)$ is determined from:

$$i(B_{12}+B_{66})q_k s_k = B_{11} - \rho c^2 - B_{66}s_k^2 \tag{12}$$

and the constants A_k are determined from the boundary condition (7). Further, if a Rayleigh-edge wave exists, then its velocity is a solution of the equation:

$$(B_{66} - \rho c^2)[B_{12}^2 - B_{22}(B_{11} - \rho c^2)] + \rho c^2\sqrt{B_{22}B_{66}}\sqrt{(B_{11} - \rho c^2)(B_{66} - \rho c^2)} \tag{13}$$

which satisfies:

$$0 < \rho c^2 < \min\{B_{11}, B_{66}\} \tag{14}$$

Equation (13) is the secular equation of the wave. We note that Chadwick [8] has shown that for $B_{11}, B_{22}, B_{66}, B_{12}$ satisfying (4), the conditions (13) and (14) are sufficient for the unique existence of Rayleigh-edge waves.

Remark 1.

i) It is clear that the secular equation (13) is much more simple than the secular equations (13) in Refs [1, 2] and (21) in Ref. [1] obtained recently by Cerv [1] and Cerv et al. [2].

ii) The secular equation (13) is valid for any orthotropic elastic materials.

2.2 An exact formula for the velocity

According to Vinh and Ogden [7], if a Rayleigh-edge wave exists, then its velocity is determined by:

$$x = \rho c^2 / B_{66} = \sqrt{b_1}b_2 b_3 \Big/ \left[(\sqrt{b_1}/3)(b_2 b_3 + 2) + \sqrt[3]{R + \sqrt{D}} + \sqrt[3]{R - \sqrt{D}} \right] \tag{15}$$

where $b_1 = B_{22}/B_{11},\ b_2 = 1 - B_{12}^2/(B_{11}B_{22}),\ b_3 = B_{11}/B_{66},\ R$ and D are given by:

$$\begin{aligned} R &= -\frac{1}{54}\, h(b_1, b_2, b_3) \\ D &= -\frac{1}{108}\left[2\sqrt{b_1}(1 - b_2)\, h(b_1, b_2, b_3) + 27b_1(1 - b_2)^2 + b_1(1 - b_2 b_3)^2 + 4 \right] \end{aligned} \tag{16}$$

in which:

$$h(b_1, b_2, b_3) = \sqrt{b_1}\ [2b_1(1 - b_2 b_3)^3 + 9(3b_2 - b_2 b_3 - 2)] \tag{17}$$

and the roots in (15) taking their principal values. It is clear that the squared dimensionless Rayleigh wave velocity x is a continuous function of three dimensionless parameters b_1, b_2, b_3

which must be satisfied the inequalities: $b_1 > 0$; $b_2 > 0$; and $b_3 > 0$ according to (4). On view of (3), three dimensionless parameters b_k are expressed in terms of E_1, E_2, G_{12}, ν_{12} as:

$$b_1 = \frac{E_2}{E_1},\ b_2 = 1 - \frac{E_2 \nu_{12}^2}{E_1},\ b_3 = \frac{E_1^2}{G_{12}(E_1 \nu_{12}^2 - E_2)} \tag{18}$$

It would be worth to note that for some other elastic media, the exact formulas for the Rayleigh wave velocity have been derived recently, see Refs. [9, 10, 11], for examples.

2.3 Approximate formulas for the velocity

On view of (14), Eq. (13) is equivalent to the equation:

$$\sqrt{B_{66} - X}\left(B_{22}X - \delta_{12}\right) + X\sqrt{B_{22}B_{66}}\sqrt{B_{11} - X} = 0 \tag{19}$$

where $X = \rho c^2$, $\delta_{12} = B_{11}B_{22} - B_{12}^2$. After squaring and rearranging Eq. (19) becomes:

$$B_{22}(B_{22} - B_{66})X^3 + B_{22}(B_{11}B_{66} - B_{22}B_{66} - 2\delta_{12})X^2 + \delta_{12}(\delta_{12} + 2B_{22}B_{66})X - B_{22}\delta_{12}^2 = 0 \tag{20}$$

Dividing two sides of Eq. (20) by $(B_{66})^5\ (>0)$ leads to:

$$m_3 x^3 + m_2 x^2 + m_1 x + m_0 = 0,\ 0 < x = X / B_{66} < 1 \tag{21}$$

where:

$$m_3 = b_1 b_3(1 - b_1 b_3),\ m_2 = b_1 b_3^2(1 - b_1 - 2b_1 b_2 b_3),\ m_1 = b_1^2 b_2 b_3^3(b_2 b_3 + 3),\ m_0 = -b_1^2 b_2^2 b_3^4 \tag{22}$$

According to Vinh and Malischewsky [4], in the interval [0, 1], the best approximate second-order polynomial of x^3 in the sense of least squares is:

$$1.5x^2 - 0.6x + 0.05 \tag{23}$$

Introducing (23) into (21) yields a quadratic equation, namely:

$$(m_2 + 1.5m_3)x^2 - (0.6m_3 - m_1)x + m_0 + 0.05m_3 = 0 \tag{24}$$

whose solution corresponding to the Rayleigh-edge wave is:

$$x = \frac{B - \sqrt{B^2 - 4AC}}{2A} \tag{25}$$

where:

$$\begin{aligned} A &= b_1 b_3[b_3(1 + 0.5b_1 - 2b_1 b_2 b_3) - 1.5], \\ B &= b_1 b_3[0.6(b_1 b_3 - 1) - b_1 b_2 b_3^2(b_2 b_3 + 2)], \\ C &= 0.05 b_1 b_3(b_1 b_3 - 1) - b_1^2 b_2^2 b_3^4. \end{aligned} \tag{26}$$

If we use the best approximate second-order polynomial of x^3 in the space $C[0,1]$, namely: $1.5x^2 - 0.5625x + 0.03125$ (see [4]), then x is given by (25) in which:

$$\begin{aligned} A &= b_1 b_3[b_3(1 + 0.5b_1 - 2b_1 b_2 b_3) - 1.5], \\ B &= b_1 b_3[0.5625(b_1 b_3 - 1) - b_1 b_2 b_3^2(b_2 b_3 + 2)], \\ C &= 0.03125 b_1 b_3(b_1 b_3 - 1) - b_1^2 b_2^2 b_3^4. \end{aligned} \tag{27}$$

As a check, we employ the obtained formulas for calculating $\sqrt{x} = c_R / c_T$ for the materials mentioned in [1] (see Figures 4, 5, 6, 12, 14, 16 in [1]), where c_R is the velocity of Rayleigh-edge waves, $c_T = \sqrt{B_{66} / \rho}$. By x_e, x_{a1}, x_{a2} and x_{cerv} we denote the value of x that is calculated respectively by the exact formula (15), by the approximate formula (25), (26), by approximate formula (25), (27) and by Cerv [1] by solving the secular equations (13), (21) in Ref. [1]. The results are presented in Table 1. Note that for given $E_1, E_2, G_{12}, \nu_{12}$, the dimensionless parameters b_1, b_2, b_3 are calculated by (18). It is shown from the table 1 that the approximations (25), (26) and (25), (27) are good ones. *With the help of the exact formula* (15) *or the approximate formulas* (25), (26) *and* (25), (27), *the study of the effect of material properties on the Rayleigh-edge wave velocity becomes much more easy.*

Material	Kevlar-Epoxy (Aramid)	Wolfram (crystal)	Fibredux - 914-C	α-Fe	Silicon	Crystal line Gold
E_1 (GPa)	87	423.1	115	156.48	141.41	53.48
E_2 (GPa)	5.5	423.1	8.85	156.48	141.41	53.48
G_{12} (GPa)	2.2	152	5.08	116	79.6	42
ν_{12} (GPa)	0.34	0.396	0.287	0.597	0.386	0.844
$\rho\,(\mathrm{kg\,m^{-3}})$	1380	19300	1560	7875	2330	19300
$\sqrt{x_e}$	0.9949	0.9244	0.9873	0.7633	0.8414	0.7321
$\sqrt{x_{a1}}$	0.9949	0.9238	0.9873	0.7613	0.8387	0.7308
$\sqrt{x_{a2}}$	0.9949	0.9228	0.9873	0.7608	0.8377	0.7305
$\sqrt{x_{cerv}}$	0.995	0.924	0.987	0.76	0.84	0.73

Table 1. Values of $\sqrt{x} = c_R / c_T$ $(c_T = \sqrt{B_{66} / \rho})$ corresponding to the materials mentioned in Ref. [1]: $\sqrt{x_e}$ calculated by the exact formula (15), $\sqrt{x_{a1}}$ calculated by the approximate formula (25), (26), $\sqrt{x_{a2}}$ calculated by approximate formula (25), (27), $\sqrt{x_{cerv}}$ obtained by Cerv by solving the secular equations (13), (21) in Ref. [1].

3 Non-principal Rayleigh-edge waves in thin orthotropic media

3.1 Basic equations

Consider a thin homogeneous orthotropic elastic panel occupying the half-space $x_2 \geq 0$ whose principal material axes are X, Y, Z. Suppose that the Z-axis coincides with the x_3-axis and coordinate system (x_1, x_2) is the rotated one from (X, Y) by counter clockwise angle θ (see Fig. 1). Suppose that the panel is subjected to the plane stress state (1). The stress-strain relation in principal material axes X, Y has the form [2, 3]:

$$\sigma_{XX} = B_{11}\varepsilon_{XX} + B_{12}\varepsilon_{YY},\ \sigma_{YY} = B_{12}\varepsilon_{XX} + B_{22}\varepsilon_{YY},\ \sigma_{XY} = 2B_{66}\varepsilon_{XY} \tag{28}$$

where B_{ij} are given by (3) in which $E_1, E, \nu_{12}, \nu_{21}, G_{12}$ are understood as $E_X, E_Y, \nu_{XY}, \nu_{YX}, G_{XY}$, respectively. In the x_1, x_2 coordinate system it holds [2, 3]:

$$\begin{aligned} \sigma_{11} &= Q_{11}\varepsilon_{11} + Q_{12}\varepsilon_{22} + 2Q_{16}\varepsilon_{12} \\ \sigma_{22} &= Q_{12}\varepsilon_{11} + Q_{22}\varepsilon_{22} + 2Q_{26}\varepsilon_{12} \\ \sigma_{12} &= Q_{16}\varepsilon_{11} + Q_{26}\varepsilon_{22} + 2Q_{66}\varepsilon_{12} \end{aligned} \tag{29}$$

where [3]:

$$\begin{aligned}
Q_{11} &= B_{11}c_\theta^4 + 2(B_{12}+2B_{66})c_\theta^2 s_\theta^2 + B_{22}s_\theta^4,\\
Q_{22} &= B_{11}s_\theta^4 + 2(B_{12}+2B_{66})c_\theta^2 s_\theta^2 + B_{22}c_\theta^4,\\
Q_{12} &= (B_{11}+B_{22}-4B_{66})c_\theta^2 s_\theta^2 + B_{12}(c_\theta^4+s_\theta^4),\\
Q_{66} &= (B_{11}+B_{22}-2B_{12}-2B_{66})c_\theta^2 s_\theta^2 + B_{66}(c_\theta^4+s_\theta^4),\\
Q_{16} &= -(B_{11}-B_{12}-2B_{66})c_\theta^3 s_\theta - (B_{12}-B_{22}+2B_{66})c_\theta s_\theta^3,\\
Q_{26} &= -(B_{11}-B_{12}-2B_{66})c_\theta s_\theta^3 - (B_{12}-B_{22}+2B_{66})c_\theta^3 s_\theta
\end{aligned} \tag{30}$$

in which $c_\theta := \cos\theta,\ s_\theta := \sin\theta\ \ (0 \le \theta \le \pi)$ and the strain ε_{ij} are expressed in terms of the displacement gradients $u_{m,n}$ as:

$$\varepsilon_{11} = u_{1,1},\ \varepsilon_{22} = u_{2,2},\ \varepsilon_{12} = (u_{1,2}+u_{2,1})/2 \tag{31}$$

In the absence of body forces, equations of motion are [3]:

$$\sigma_{11,1} + \sigma_{12,2} = \rho \ddot{u}_1,\ \sigma_{12,1} + \sigma_{22,2} = \rho \ddot{u}_2 \tag{32}$$

Following the same procedure carried out in [12, Section 2], from Eqs. (29), (31) and (32) we have:

$$\begin{bmatrix} u' \\ \sigma' \end{bmatrix} = N \begin{bmatrix} u \\ \sigma \end{bmatrix} \tag{33}$$

where $u = [u_1, u_2]^T$, $\sigma = [\sigma_{12}, \sigma_{22}]^T$, the symbol T indicates the transpose of matrices, the prime indicates differentiation with respect to x_2 and:

$$\begin{aligned}
&N = \begin{bmatrix} N_1 & N_2 \\ K & N_3 \end{bmatrix},\quad N_1 = \begin{bmatrix} (d_1/d)\partial_1 & -\partial_1 \\ -(d_2/d)\partial_1 & 0 \end{bmatrix},\quad N_2 = \frac{1}{d}\begin{bmatrix} Q_{22} & -Q_{26} \\ -Q_{26} & Q_{66} \end{bmatrix},\\
&K = \begin{bmatrix} \rho\partial_t^2 - (d_3/d)\partial_1^2 & 0 \\ 0 & \rho\partial_t^2 \end{bmatrix},\quad N_3 = N_1^T
\end{aligned} \tag{34}$$

Here we use the notations: $\partial_1 = \partial/(\partial x_1), \partial_1^2 = \partial^2/(\partial x_1^2), \partial_t^2 = \partial^2/(\partial t^2)$ and:

$$d = Q_{22}Q_{66} - Q_{26}^2,\ d_1 = Q_{12}Q_{26} - Q_{22}Q_{16},\ d_2 = Q_{12}Q_{66} - Q_{16}Q_{26},\ d_3 = Q_{11}d + Q_{16}d_1 - Q_{12}d_2 \tag{35}$$

In addition to Eq. (33), the displacement vector u and the traction vector σ are required to satisfy the decay condition at the infinity:

$$u(+\infty) = 0,\ \sigma(+\infty) = 0 \tag{36}$$

and the free-traction condition at the edge $x_2 = 0$:

$$\sigma(0) = 0 \tag{37}$$

3.2 Explicit secular equation

Now we consider the propagation of a Rayleigh wave, travelling with velocity c and wave number k in the x_1-direction. The components u_1, u_2 of the displacement vector and σ_{12}, σ_{22} of the traction vector at the planes $x_3 = \text{const}$ are found in the form:

$$\{u_1, u_2, \sigma_{12}, \sigma_{22}\}(x_1, x_2, t) = \{U_1(kx_2), U_2(kx_2), ik\Sigma_1(kx_2), ik\Sigma_1(kx_2)\}\mathrm{e}^{ik(x_1-ct)} \tag{38}$$

Substituting (38) into (33) yields:

$$\begin{bmatrix} U' \\ \Sigma' \end{bmatrix} = iM \begin{bmatrix} U \\ \Sigma \end{bmatrix} \tag{39}$$

where $U = [U_1\, U_2]^T$, $\Sigma = [\Sigma_1\, \Sigma_2]^T$, and:

$$M = \begin{bmatrix} M_1 & M_2 \\ Q & M_3 \end{bmatrix}, \quad M_1 = \begin{bmatrix} d_1/d & -1 \\ -d_2/d & 0 \end{bmatrix}, \quad M_2 = \frac{1}{d}\begin{bmatrix} Q_{22} & -Q_{26} \\ -Q_{26} & Q_{66} \end{bmatrix},$$
$$Q = \begin{bmatrix} X - d_3/d & 0 \\ 0 & X \end{bmatrix}, \quad M_3 = M_1^T \tag{40}$$

the prime in Eq. (39) indicates differentiation with respect to $y = kx_2$. From(40), one can see that the characteristic equation $|M - pI| = 0$ of Eq. (39) is a fully quartic equation for p (see also [3]), therefore the explicit secular equation of the wave could not be derived by the traditional approach. In order to obtain it we will employed the method of first integrals [5, 6]. Eliminating U from(39), we have:

$$\alpha\Sigma'' - i\beta\Sigma' - \gamma\Sigma = 0 \tag{41}$$

where the matrices $\alpha,\ \beta,\ \gamma$ are given by:

$$\alpha = Q^{-1} = \begin{bmatrix} \dfrac{d}{dX - d_3} & 0 \\ 0 & \dfrac{1}{X} \end{bmatrix} \tag{42}$$

$$\beta = M_1 Q^{-1} + Q^{-1} M_3 = \begin{bmatrix} \dfrac{2d_1}{dX - d_3} & -\dfrac{1}{X} - \dfrac{d_2}{dX - d_3} \\ -\dfrac{1}{X} - \dfrac{d_2}{dX - d_3} & 0 \end{bmatrix} \tag{43}$$

$$\gamma = M_1 Q^{-1} M_3 - M_2 = \begin{bmatrix} \dfrac{d_1^2}{d(dX - d_3)} + \dfrac{1}{X} - \dfrac{Q_{22}}{d} & -\dfrac{d_1 d_2}{d(dX - d_3)} + \dfrac{Q_{26}}{d} \\ -\dfrac{d_1 d_2}{d(dX - d_3)} + \dfrac{Q_{26}}{d} & \dfrac{d_2^2}{d(dX - d_3)} - \dfrac{Q_{66}}{d} \end{bmatrix} \tag{44}$$

Note that α, β, γ are symmetric real matrices. From (36)-(38) it follows:

$$\Sigma(0) = \Sigma(+\infty) = 0 \tag{45}$$

It is not difficult to verify that from (41) and (45) it follows (see also [6]):

$$\begin{vmatrix} \alpha_{11} & \beta_{11} & \gamma_{11} \\ \alpha_{12} & \beta_{12} & \gamma_{12} \\ \alpha_{22} & \beta_{22} & \gamma_{22} \end{vmatrix} = 0 \tag{46}$$

Introducing (42)-(44) into (46) yields:

$$\begin{aligned} F(X,\theta) \equiv {} & dX^2[(d + d_2)X - d_3][d_2^2 - Q_{66}(dX - d_3)] \\ & + (dX - d_3)[(d + d_2)X - d_3][Q_{22}dX^2 - (d^2 + d_1^2 + Q_{22}d_3)X + dd_3] \\ & - 2d_1 X^2 (dX - d_3)[Q_{26}(dX - d_3) - d_1 d_2] = 0 \end{aligned} \tag{47)}$$

Equation (47) is the desired explicit secular equation and it is a quartic equation for $X = \rho c^2$. It is not difficult to verify that

$$c_R(\theta) = c_R(\pi - \theta) \text{ and if } B_{11} = B_{22} \text{ then } c_R(\theta) = c_R(\pi/2 - \theta) \tag{48}$$

where c_R is the velocity of Rayleigh-edge waves. *The explicit secular equations* (47) *is a convenient tool for analyzing the effect of the orientation of principal material directions as well as of the material properties on the velocity of Rayleigh-edge waves. They can also be used to solve the inverse problems: determining the material parameters from measured values of the velocity.*

4 Conclusions

In this paper we arrive at a secular equation for principal Rayleigh-edge waves that is valid for all orthotropic elastic materials and much more simple than the ones obtained recently by Cerv [1] and Cerv et al. [2]. Exact and approximate formulas for the velocity of principal Rayleigh-edge waves are also established and they are a powerful tool for analyzing the effect of material parameters on the Rayleigh-edge wave velocity. For non-principal Rayleigh-edge waves a secular equation in explicit form is obtained by using the method of first integrals.

Acknowledgments

The work was supported by the Vietnam National Foundation For Science and Technology Development (NAFOSTED).

References

[1] Cerv J. The Influence of Material Properties on the Behaviour of Rayleigh Edge Waves in Thin Orthotropic Media. *Int Rev Mech Eng* (IREME), **2**, pp. 762-72 (2008).

[2] Cerv J, Kroupa T, and Trnka J. Influence of principal material directions of thin orthotropic structures on Rayleigh-edge wave velocity. *Composite Structures*, **92**;, pp. 568-77 (2010).

[3] Ohyoshi T. The propagation of Rayleigh waves along an obliquely cut surface in a directional fibre-reinforced composite. *Compos. Sci. Technol.*, **60,** pp**.** 2191-2196 (2000).

[4] Pham Chi Vinh, Malischewsky P. An approach for obtaining approximate formulas for the Rayleigh wave velocity. *Wave Motion*, **44**, pp. 549-562 (2007).

[5] Mozhaev V G. *Some new ideas in the theory of surface acoustic waves in anisotropic media.* In “IUTAM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solid Mechanics”, edited by Parker D F and England A H, Kluwer Acad Publ; 1995, pp. 455-462.

[6] Destrade M. The explicit secular equation for surface acoustic waves in monoclinic elastic crystals. *J. Acous.t Soc. Am*., **109**; pp. 1398-402 (2001).

[7] Pham Chi Vinh, Ogden R W. Formulas for the Rayleigh wave speed in orthotropic elastic solids. *Arch. Mech.,* **56**, pp. 247-65 (2004).

[8] Chadwick P. The existence of pure surface modes in elastic materials with orthohombic symmetry. *J. Sound Vibr.,* **47**, pp. 39-52 (1976).

[9] Ogden R. W. and Pham Chi Vinh. On Raylegh waves in incompressible orthotropic elastic solids. *J. Acous.t Soc. Am.* , **115**, pp. 530-3 (2004).

[10] Pham Chi Vinh. On Formulas for the velocity of Rayleigh waves in prestrained incompressible elastic solids. *Trans. ASME J. Appl. Mech.*, 77; 021006 (9 pages) (2010).

[11] Pham Chi Vinh. On formulas for the Rayleigh wave velocity in pre-stressed compressible solids. *Wave Motion*, **48**;, pp. 613-24 (2011).

[12] Pham Chi Vinh. Explicit secular equations of Rayleigh waves in elastic media under the influence of gravity and initial stress. *Appl. Math. Comput.*, **215,** pp. 395-404 (2009).