IDENTIFICATION OF ELASTIC PROPERTIES FROM FULL-FIELD MEASUREMENTS APPLYING THE MODIFIED CONSTITUTIVE RELATION ERROR STRATEGY

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Abstract
The aim of this paper is to present an inverse approach dedicated to the exploitation of full-field measurements, to identify elastic properties of heterogeneous materials, such as composites, in the static case. The method is based on the modified constitutive relation error principle and could be split in two steps. The first one consists in defining mechanical fields from the available theoretical and experimental data, for a fixed set of mechanical parameters, by the minimization of a criterion allowing a compromise between constitutive equation and measurements adequacy. The second step takes the form of minimizing a cost function defined by using these fields, to identify the sought material properties. Moreover, the robustness of the method was tested on some numerical examples where white Gaussian perturbations were added to the displacement field to simulate an experimental errors.

1 Introduction

The intensive use of composite materials in aeronautic industry requires the knowledge of the parameters governing the mechanical constitutive material behavior. When focusing on their elastic behavior, composite materials have generally heterogeneous properties. It is then necessary to characterize them in a reliable way. Full-field measurements are often needed to exploit heterogeneous tests, hardly possible with classic measure tools like extensometric gages which provide a limited number of measurement points. It leads to the development of digital images correlation (DIC) techniques [1, 2, 3], which constitute one of the main breakthroughs of the last 30 years. Thus, it is necessary to develop identification strategies adapted to the fullness of this type of measurements, and based on inverse approaches.
Besides, measurement perturbations, which are inherent in experimental tests, affect the accuracy of the identified properties, especially in elasticity, when the magnitude of strain fields is relatively low. Thus, it is important to take them into account and use a strategy which is less dependent on perturbations.
Numerous identification methods have been proposed and adapted to the use of full-field measurements. A review of the most widespread methods (Finite Element Model Updating (FEMU), Virtual Fields Method, Equilibrium Gap Method, Reciprocity Gap Method and Constitutive Relation Error (CRE)) can be found in [4]. Among the identification procedures, the so-called modified constitutive relation error (MCRE) turns out as a robust method towards measurements perturbations [5]. At first, it was used for model updating in vibration dynamics within a validation framework [6, 7]. Then, the method was adapted to the exploitation of full-field measurements in [8, 9, 10].

The present paper presents a formulation of an identification problem based on the MCRE, leading to the minimization of a cost function composed of two parts. One related to the constitutive equation and another to the distance between the simulation and the experimental measurements.
An important point to be noted is that the modeling of the boundary conditions represents a shortcoming of some inverse methods. The present formulation allows identification even if they are not available. No hypothesis on boundary conditions is thus required, for example to take into account the distribution of the load on a sample boundary.
In the following, we present the identification framework and describe the various steps of the MCRE.
The description of the numerical implementation of the presented formulation and the adopted resolving algorithm will be discussed. Identification results will be presented and the robustness of the method will be illustrated on a numerical case.

2 The modified constitutive relation error

2.1 Identification framework

Let us consider a structure subjected to a mechanical load, defined in a domain Ω and governed by the equations of continuum mechanics. Here, we are looking for its elastic properties noted Θ. In the following formulation, We make the hypothesis that boundary conditions are unknown, and one disposes only of displacement measurements noted $\underline{\tilde{u}}$, available on a given part Ω_m of the entire domain, as shown in figure 1. Thus, the mechanical equations are:

$$\begin{cases} \text{Equilibrium:} & div\,\underline{\underline{\sigma}} = 0 & \text{in } \Omega & \text{(1a)} \\ \text{Kinematic compatibility:} & \underline{\underline{\varepsilon}} = \frac{1}{2}(\nabla \underline{u} + \nabla' \underline{u}) & \text{in } \Omega & \text{(1b)} \\ \text{Constitutive equation:} & \underline{\underline{\sigma}} = \mathbb{C}(\Theta)\,\underline{\underline{\varepsilon}} & \text{in } \Omega & \text{(1c)} \\ \text{Measurements:} & \underline{u} = \underline{\tilde{u}} & \text{in } \Omega_m & \text{(1d)} \end{cases}$$

with $\mathbb{C}$ the stiffness tensor, $\underline{u}$ the displacement vector, $\underline{\underline{\sigma}}$ the stress tensor and $\underline{\underline{\varepsilon}}$ the strain tensor.

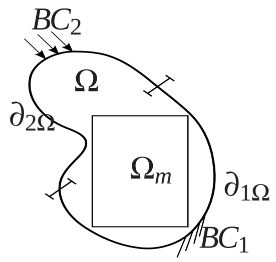


Figure 1. Refrence problem related to the identification problem

Ω_m is a non-zero measurement subdomain, defined as $\Omega_m \subset \Omega$. Measurements are stored in vector $\underline{\tilde{u}}$, defined as: $\underline{\tilde{u}} \in \mathbb{R}^{2\times n}$ in a 2D problem, with n the number of measurement points.
The formulation of the identification problem, based on the so-called modified constitutive relation error (MCRE) is a variational inverse approach split in two steps. The first step consists in building admissible stress and strain fields for a fixed set of mechanical parameters Θ, by using the whole experimental and theoretical information in our disposal. This step defines the reference problem. The second one consists in finding the best model's parameters Θ by minimizing a cost function derived from the solution fields of the previous step.

2.2 The basic problem

One of the MCRE's principles relies on the partitioning of the previous equations into reliable and unreliable equations. Then, during the identification process, the method consists in the exact verification of the equations which are considered as reliable, and the relaxation of uncertain relations, which are verified at best by a weighted minimization of the MCRE functional. The two relation's sets are defined as:

1. *Reliable equations* : mechanical equilibrium (1a) and kinematic compatibility (1b).
2. *Uncertain equations* :
   - the constitutive equation (1c), containing the sought behavior properties .
   - measurements (1d) , because they are spotted by perturbations. This point represents the difference between MCRE and CRE methods and allots the adjective *Modified* to the presented procedure.

This separation leads to the following formulation of the identification problem, which defines the reference problem for a fixed parameters Θ:

Find the couple $(\underline{u},\underline{v})$ that minimizes:

$$\mathcal{J}(\underline{u},\underline{v}) = \underbrace{\frac{1}{2}\int_{\Omega}\underline{\underline{\sigma}}(\underline{v}-\underline{u}) : \underline{\underline{\varepsilon}}(\underline{v}-\underline{u})d\Omega}_{\mathcal{J}_1} + \alpha\underbrace{\frac{1}{2}\int_{\Omega_m}(\underline{u}-\underline{\tilde{u}})^2 d\Omega_m}_{\mathcal{J}_2} \tag{2}$$

under reliable equations constraints (1a) and (1b). Where $\underline{u}$ and $\underline{v}$ are the sought admissible fields. The solution of the basic problem is denoted as:

$$\begin{cases} (\underline{u}(\Theta),\underline{v}(\Theta)) = \arg\min\limits_{\underline{u},\underline{v}} \mathcal{J}(\underline{u},\underline{v}) \\ \text{under constraints (1a) and (1b)} \end{cases}$$

Remark. $\underline{v}$ is a statically admissible field. Actually, the presented formulation is written in a displacement way and one can show that: $\exists\, \underline{v}$ such as $\underline{\underline{\sigma}} = \mathbb{C}\, \underline{\underline{\varepsilon}}(\underline{v})$, where $\mathbb{C}$ is the Hooke's tensor.
The functional $\mathcal{J}$ is composed of two parts. A first component $\mathcal{J}_1$ relative to the specific constitutive equation, and a second one $\mathcal{J}_2$ expressing the gap between simulated and measured fields. Its minimization leads to the resolution of a linear system, and then, the achievement of the admissible fields building (see section 3.1).
α is an hyperparameter used to make the functional terms have the same unit, and plays also the role of a weighting factor allowing to balance their magnitudes.

2.3 The identification problem

The solving of the basic problem (2)yields the admissible fields for any set of parameters and one can now proceed seeking the parameters Θ, thanks to the minimization of a cost function $\mathcal{G}$ which has the same form of the functional $\mathcal{J}$ but is function of the optimal fields $\underline{u}$ and $\underline{v}$. Identification problem is then written as:

$$\begin{aligned} \Theta &= \arg\min_{\Theta'} \mathcal{G}(\Theta') \\ \text{with} \quad \mathcal{G}(\Theta') &= \mathcal{J}(\underline{u}(\Theta'),\underline{v}(\Theta')) \end{aligned}$$

3 Numerical implementation

3.1 Discretization of the basic problem

To implement the presented method numerically, one should discretize of the continuous problem (2). The choice here is to use the finite element method. Thus, after the meshing of the domain Ω, solution fields are represented by discrete vectors containing displacement on the degrees of freedom, denoted U and V. The column vector $\tilde{U}$ stores measured displacements available on the measurement grid and K denotes the discrete rigidity matrix.
In order to solve the numerical problem, the discrete formulation will be based on the partitioning of mesh nodes into two groups: internal nodes, denoted by the index "i", and border nodes denoted by the index "b". When dealing with both internal and border nodes, the index "." is used. Thus,

the problem (2) takes the following discrete form:
Find (U,V) that minimize:

$$\begin{cases} \mathcal{J}(U,V) = \underbrace{\frac{1}{2}(U-V)^T K(U-V)}_{\mathcal{J}_1} + \alpha \underbrace{\frac{1}{2}(\Pi U - \tilde{U})^T (\Pi U - \tilde{U})}_{\mathcal{J}_2} & \text{(4a)} \\ \text{under the constraint} \quad K_{i.}V = 0 & \text{(4b)} \end{cases}$$

One can remark that the equilibrium constraint of the reference problem stands only for the internal nodes. Thus, all the columns of the rigidity matrix are concerned. In order to evaluate the distance-to-measurements term, a discrete operator Π is introduced and plays two roles: extracting an adequate size of the kinematic admissible field U defined on the whole domain, to fit the subdomain Ω_m if smaller than Ω, and projecting that extracted part on the measurement grid.
To simplify mathematical writing, we make hypothesis that Π is only applicable on the internal nodes of the mesh. Finally, functional $\mathcal{J}$ is minimized under the constraint (4b). Thus, a Lagrange multiplier Λ is introduced:

$$\mathcal{L}(U,V,\Lambda_i) = \mathcal{J}(U,V) - \Lambda_i^T (K_{i.}V)$$

When studying the stationarity of the Lagrangian, we obtain three equations $\left(\frac{\partial \mathcal{L}}{\partial U}, \frac{\partial \mathcal{L}}{\partial V}, \frac{\partial \mathcal{L}}{\partial \Lambda}\right) = (0,0,0)$. After doing the partitioning on both internal and border nodes, we obtain the following simultaneous equations:

$$\begin{cases} (K_{ii} + \alpha \Pi^T \Pi) U_i + K_{ib} U_b = \alpha \Pi^T \tilde{U} & \text{(5a)} \\ K_{bi} U_i + K_{bb} U_b - K_{bi} V_i - K_{bb} V_b = 0 & \text{(5b)} \\ K_{ii} U_i + K_{ib} U_b = K_{ii}^T \Lambda_i & \text{(5c)} \\ K_{bi}(U_i - V_i) + K_{bb}(U_b - V_b) = K_{bi} \Lambda_i & \text{(5d)} \\ K_{ii} V_i + K_{ib} V_b = 0 & \text{(5e)} \end{cases}$$

3.2 Solving algorithm

3.2.1 Discrete reference problem

From equations (5c) and (5d), one can easily show that $(U_b - V_b)$ corresponds to the degrees of freedom of a rigid body motion on the domain border. Besides, one can show that functional $\mathcal{J}$ is insensitive to that rigid body motion, that is why it is chosen equal to zero ($U_b = V_b$).
Thereby, equations (5b) and (5e) lead to a linear system as a function of U:

$$\begin{bmatrix} K_{ii} + \alpha \Pi^T \Pi & K_{ib} \\ K_{bi} & K_{bi} K_{ii}^{-1} K_{ib} \end{bmatrix} \begin{bmatrix} U_i \\ U_b \end{bmatrix} = \begin{bmatrix} \alpha \Pi^T \tilde{U} \\ 0 \end{bmatrix}$$

By substituting U_i with U_b, U_b is deduced as a solution of the following linear system:

$$\underbrace{K_{bi} \left[(K_{ii} + \alpha \Pi^T \Pi)^{-1} - K_{ii}^{-1} \right] K_{ib}}_{M} U_b = \underbrace{K_{bi} (K_{ii} + \alpha \Pi^T \Pi)^{-1} \alpha \Pi^T \tilde{U}}_{F} \qquad (6)$$

Let us note that M becomes a rank-deficient matrix when Ω_m is smaller than Ω. In that case, we choose a QR decomposition, where Q is an orthogonal matrix and R an upper triangular matrix, such as:

$$M = QR = [Q_1 \quad Q_2] \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$$

The solution of problem (6) has a particular solution and a second part relative to the kernel of M. Since the functional's magnitude is insensitive to the kernel member choice, it is chosen equal to zero. After the *QR* decomposition, the problem solution is:

$$U_b = R_1^{-1} Q_1^T F \tag{7}$$

The other quantities $(U_i, V_i \text{ and } V_b)$ are simply deduced from U_b. Next step is the identification of the material properties by the minimization of the cost function $\mathcal{G}\left(U(\Theta^{'}), V(\Theta^{'})\right)$.

3.2.2 Calculation of the gradient

A great specificity of the present procedure is the use of the same functional to describe both reference and identification problems. The interest of that particularity is an immediate access to the expression of the cost function gradient, very helpful for the minimization step. Generally, when the gradient expression is not directly available, it can be computed by using adjoint state method. In the present formulation, the reference problem and the adjoint state are coupled and solved together. Here, the cost function is equal to the Lagrangian At the solution fields U, V, Λ_i, the cost function is equal to the Lagrangian. To calculate the gradient one just needs to work out partial derivatives of the Lagrangian with respect to material properties, that is:

$$\frac{\partial \mathcal{G}}{\partial \Theta_k^{'}} = \frac{\partial \mathcal{L}}{\partial \Theta_k^{'}} = \frac{1}{2}(U-V)^T \frac{\partial K}{\partial \Theta_k^{'}}(U-V) - \Lambda_i^T \frac{\partial K_{i.}}{\partial \Theta_k^{'}} V \tag{8}$$

where $\Theta_k^{'}$ denotes the k^{th} sought material property.

4 Application to numerical examples

In this section, we illustrate the method through two numerical examples where data are numerically built. Random perturbations are mixed with the constructed displacement field, in order to simulate experimental errors. To perform the identification by the MCRE, one only has to solve the basic problem to build the admissible displacement fields, and then use an adequate minimization algorithm, depending on the number of sought properties.

4.1 Test case 1

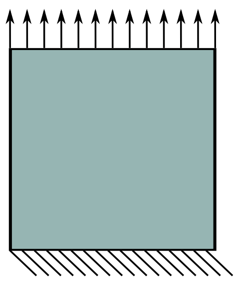
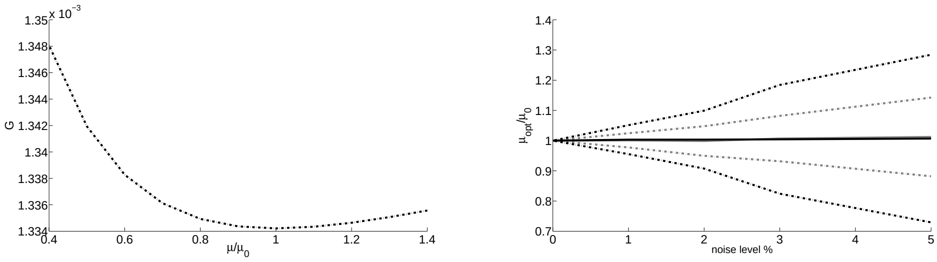


Figure 2. First example being studied

The first test case being studied (Figure 2) is an elastic square plate, clamped on one side and subjected to uniformly distributed load on the opposite side. The plate is then meshed by Q4-elements and considered elastic and homogeneous. Measurements are artificially built with given reference values of Lame coefficients λ_0 and μ_0. Rather than λ, we will identify μ since it is more sensitive to the displacement field here.

In order to evaluate the robustness of the method, the dependence of the cost function on the perturbation magnitude will be studied. Figure 3(a) shows the shape of the cost function when perturbed with a white Gaussian noise whose magnitude is equal to 5% of the mean value of the displacements. Its minimum is still well found.

Figure 3(b) presents the identification results on 700 samples of perturbation, in terms of mean value and standard deviation for the μ modulus as a function of the noise level. Two MCRE



(a) Cost function for perturbed measurements (b) Standard deviation

Figure 3. Cost function and identification results

formulations are compared, the presented one (in black) without any knowledge on the boundary conditions, and a second formulation (in gray) taking into account the knowledge of the free-edges ($\underline{\underline{\sigma}}.\underline{n} = \underline{0}$) as a reliable information. Such an information improves the identification results.

4.2 Test case 2

The second test case is an elastic isotropic plate subjected to a 3 points bending. The plate is meshed with T3-elements, and we look for homogeneous Lame coefficient μ. The displacement measurements are created from a reference calculation with reference Lame coefficients (λ_0, μ_0) and are illustrated on Figure 4.

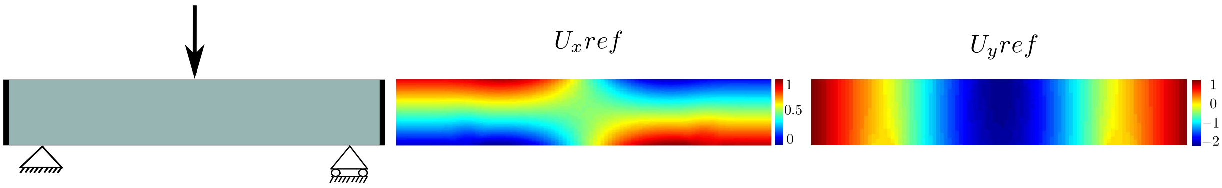
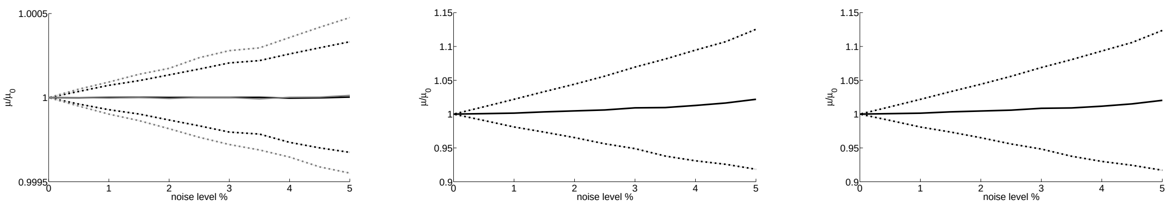


Figure 4. Simulation of a 3 bending test (geometry and displacement fields)

Besides, we propose to compare the MCRE method to the FEMU method [4] when dealing with approximately the same type of data, where perturbations are added to displacement information. The expression of the cost function of FEMU method is: $\mathcal{J} = \frac{1}{2}(\Pi U - \tilde{U})^T(\Pi U - \tilde{U})$. Since it always needs the knowledge of the boundary conditions, the size of measured zone is extended to the whole domain ($\Omega_m = \Omega$). For each data, the basic problem of each method is detailed:

1. Data 1: perturbed displacements on Ω_m and exact boundary conditions (unperturbed).
   Type of the basic problem :
   - FEMU: exact mixed boundary conditions.
   - MCRE: exact Neumann boundary conditions on $\partial_f\Omega$.
2. Data 2: perturbed displacements on Ω_m.
   Type of the basic problem :
   - FEMU: perturbed Dirichlet boundary conditions treated as reliable.
   - MCRE: no boundary conditions.
3. Data 3: perturbed displacements on Ω_m and exact free-edges information.
   Type of the basic problem :
   - FEMU: free-edges information and perturbed Dirichlet boundary conditions elsewhere, treated as reliable.
   - MCRE: free-edges information treated as reliable.

It is to be noted that, contrarily to the FEMU method, the MCRE method handles available data at its fair value. Indeed, uncertain boundary conditions are treated as reliable by the FEMU method whereas MCRE method treats as reliable only the exact information such that free-edges information, or exact boundary conditions (data 1).
In Figure5, FEMU identification results are represented by black curves when MCRE results are represented by gray ones.



(a) data 1 (b) data 2 (c) data 3

Figure 5. Comparison between FEMU and MCRE methods for different types of data

Figure 5(a) shows identification results when dealing with the true boundary conditions, which constitute an unreal case. In fact, no exact boundary conditions can be obtained during a mechanical test. One can observe how small the magnitude of the standard deviation is. Here, the FEMU method gives better results in such interval for the standard deviation.
Figures 5(b) and 5(c) present identification results of FEMU method for the last two data, where all displacement data are corrupted. The corresponding MCRE results are not ready yet but will be presented during the oral presentation. The first studied example confirmed the robustness of the method. It shows that the MCRE method is effective in the total or partial absence of BC information in the presence of corrupted measurements.

5 Conclusion

In this paper, we considered the problem of the identification of material parameters in the framework of static. An identification strategy based on the use of the modified constitutive relation error was proposed. It has the particularity of taking into account the whole theoretical and experimental data to build admissible fields, then, the identification step starts by the minimization of a cost function with regards to those optimal fields. The presented approach is formulated in a displacement way and has the advantage of being robust with respect to measurement uncertainties.
When inverting the linear system in the first step, a *QR* decomposition was proposed to avoid kernel problems occurring when the size of measurement zone is smaller than the size of the whole domain zone. The method was applied to heterogeneous materials, in the case of orthotropic elasticity. Identification of orthotropic parameters is being developed thanks to a coupling with a minimization procedure using the available gradient information.
Besides, an optimal control problem was studied, taking into account the only distance-to-measurements term of the MCRE functional. Its results will be discussed during the oral presentation.
The work prospect is to develop the MCRE strategy by taking into account loading information such as the resulting load, in the set of unreliable equations, by adding a distance-to-load term into the MCRE functional. The next step will focus on the application of this method to real full-field measurements.

References

[1] Chu T.C., Ranson W.F., Sutton M.A., Peters W.H. Applications of digital-image-correlation techniques to experimental mechanics. *Exp Mech* **25** 232–244. 1985

[2] Hild F., Roux S. Digital image correlation : from displacement measurement to identification of elastic properties - a review. *Strain* **42** 69-80. 2006

[3] A. Kobayashi. 1993 *Handbook on Experimental Mechanics, Wiley*

[4] Avril S., Bonnet M., Bretelle A-S., Grédiac M., Hild F., Ienny P., Latourte F., Lemosse D., Pagano S., Pagnacco E., Pierron F. Identification from measurements of mechanical fields. *Exp Mech* **48** 381-402. 2008

[5] Feissel P., Allix O. Modified constitutive relation error identification strategy for transient dynamics with corrupted data: The elastic case. *Computer Methods in Applied Mechanics and Engineering* **196** 1968-1983. 2007

[6] Ladevèze P., Reynier M., Maya N. Error on the constitutive relation in dynamics. *H.D. BUI M. T. . A., Ed., Inverse problems in Engineering* 251-256. 1994

[7] Calloch S., Dureisseix D., Hild F. Model validation in the presence of uncertain experimental data. *Engineering Computations* **21(8)** 808-833. 2002

[8] Deraemaeker A., Ladevèze P., Romeuf T. Identification de modèles de comportement de matériaux solides : utilisation d'essais et de calculs. *Technologies et Formations* **100** 36-41. 2002

[9] Latourte F. Identification Des Paramètres d'une loi élastoplastique de Prager et calcul de champs de contrainte dans des matériaux hétérogènes. *Ph.D thesis (Université de Montpellier 2)*. 2007

[10] Florentin E., Lubineau G. Identification of the parameters of an elastic material model using the constitutive equation gap method. *Comput Mech* **46** 521-531. 2010

[11] Claire, D., Hild, F., Roux, S. Identification of damage fields using kinematic measurements. *C. R. Mecanique* **330** 729-734. 2002

[12] Ben Azzouna M., Périé J-N., Guimard J-M., Hild F., Roux S. On the identification and validation of an anisotropic damage model by using full-field measurements. *International Journal of Damage Mechanics* **20(8)** 1130-1150. 2011

[13] Ben Azzouna M., Feissel P., Villon P. Identification à partir de mesures de champs - Application de l'erreur en relation de comportement modifiée. *10th Colloque National en Calcul des Structures - Giens.* 2011